Holomorphic curves and knot conormals

Tobias Ekholm

Uppsala University and Institut Mittag-Leffler, Sweden

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This talk reports on various joint works: with Aganagic-Vafa-Ng, Kucharski-Longhi, Shende, and Ng.

Plan:
- Ooguri-Vafa large N duality
- Skein valued open GW-invariants
- Knot contact homology and recursion
- Holomorphic disks and quivers
**Geometric setup:** $K \subset S^3$ – knot. $L_K \subset T^* S^3$ – Lagrangian conormal $\approx S^1 \times \mathbb{R}^2$. Shift $L_K$ off of 0-section $S^3$ (non-exact). Transition to resolved conifold $X = \{ \mathcal{O}(-1)^{\otimes 2} \rightarrow \mathbb{C}P^1 \}$.

\[
\{ z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0 \}
\]
Gauge: The $n^{th}$ symmetrically colored HOMFLY polynomial of $K$, $H_{K,n}(a,q)$. Witten’s definition ($A - U(N)$-connection):

$$H_{K,n}(q^N, q) = \int DA \, e^{\frac{ik}{4\pi} \int \text{tr}(A dA + \frac{2}{3} A^3)} \, \text{tr}_n \text{Hol}_K(A), \quad q = e^{\frac{2\pi i}{k+N}}$$

String: The open GW partition function $\Psi_K(e^x, a, g_s) = e^{F_K}$ counts holomorphic curves $u: \Sigma \rightarrow X$ with boundary on $L_K$ in homology class $[u(\Sigma)] = m[\mathbb{C}P^1] + n[x] \in H_2(X, L_K)$ according to

$$w(u) \, g_s^{-\chi(\Sigma)} \, a^{2m} \, e^{nx}.$$
Conjecture

The GW partition function equals the generating function for the colored HOMFLY:

\[ \Psi_K(x, a, g_s) = \sum_{k \geq 0} H_{K,n}(a, e^{g_s/2})e^{nx} \]
The framed skein module of a 3-manifold $L$, $\text{Sk}(L)$, is generated over $\mathbb{C}[q^{\pm1}, a^{\pm1}]$ by isotopy classes of framed links and (with $z = (q - q^{-1})$)

For example, $\text{Sk}(S^3) = \mathbb{C}[q^{\pm1}, a^{\pm1}]$, $\text{Sk}(S^1 \times \mathbb{R}^2)$ is a free commutative algebra on countably many generators $A_m$ ($m - 1$ crossings, $m$ times around).
Skein valued curve counts

$X$ CY3, $L \subset X$ Maslov zero Lagrangian. Define a skein valued curve count by summing over all moduli spaces of disconnected \textit{bare} holomorphic curves with boundary on $L \subset X$. A bare curve $u: (\Sigma, \partial \Sigma) \rightarrow (X, L)$ contributes

$$w(u) \ z^{-\chi(\Sigma)} \ a^{\text{lk}(L,u)} \langle u(\partial \Sigma) \rangle$$

\textbf{Auxiliary geometric data:} Morse function $f: L \rightarrow \mathbb{R}$. 4-chain $C$ with $\partial C = 2 \cdot L$ and normal $\pm J \cdot \nabla f$ along the boundary.
The skein count is invariant under deformations:

1) Constant curves bubble off only in codimension two.

2) At tangencies with $\nabla f$ a kink is traded for a 4-chain intersection.
For a small shift of the conormal there is a unique holomorphic cylinder. SFT stretching removes all boundaries from the 0-section (outside curves asymptotic to Reeb orbits of index 2 gives negative dimension). Calculating the skein valued invariant gives the colored HOMFLY (obvious for once around, for many times we use info about the unknot). Curves in the stretched structure are the same as in the conifold for small area $\mathbb{C}P^1$.\[\square\]
Holomorphic curves on $L_K \subset X$ are somewhat complicated. They however interact with curves at infinity that are easier to deal with and give information of the curves inside. At infinity, with $\Lambda_K$ the Legendrian conormal of $K$:

$$\partial_\infty (X, L_K) = (ST^*S^3, \Lambda_K).$$

We start at the semi-classical level,

$$\psi_K = \exp(g_s^{-1}W_K + W^K_0 + g_s W^K_1 + \ldots),$$

i.e., with $W_K$ counting curves with $\chi = 1$, disks.
The knot contact homology algebra, $CE(\Lambda_K)$, is the free unital (non-commutative) graded algebra

$$CE(\Lambda_K) = \mathbb{C}[H_2(ST^*S^3, \Lambda_K)] \langle \text{Reeb chords} \rangle = \mathbb{C}[e^{\pm x}, e^{\pm p}, a^{\pm 1}] \langle \text{Reeb chords} \rangle$$

The grading of Reeb chords is defined by a Maslov index. For $\Lambda_K$, Reeb chords correspond to oriented binormal geodesics on $K$. Grading equal to Morse index (in $\mathbb{R}^3$-patch, min = 0, sad = 1, max = 2).

The differential is linear, satisfies Leibniz rule, and defined on generators through a holomorphic curve count. The dg-algebra $CE(\Lambda_K)$ is invariant under deformations up to homotopy and in particular up to quasi-isomorphism.
Knot contact homology

\[ \varphi: (D, \partial D) \to (\mathbb{R} \times \mathbb{Y}, \mathbb{R} \times \Lambda), \]

\[ \partial \varphi + \mathcal{J} \cdot \partial \varphi \circ i = 0. \]

\[ \partial a = \sum \left| M_A(a;b) \right| e^A b \]

\[ \text{where } |a| - |b| = 1 \]

\[ b_1 b_2 \ldots b_k ; \quad b = b_1 \ldots b_k \]
\[ \partial^2 = 0 : \]

\[ \partial \left( \frac{1}{\cap \cap \cap} \right) = \frac{1}{\cap \cap} \]

In particular,

\[ \frac{1}{\cap \cap \cap} \]
In general, the knot contact homology can be explicitly computed from a braid presentation of a link. For a braid on \(n\) strands the algebra has \(n(n-1)\) generators in degree 0, \(n(2n-1)\) in degree 1, and \(n^2\) in degree 2.

**The unknot** \(U\)

\[
\mathbb{C}[e^{\pm x}, e^{\pm p}, a^{\pm 1}]\langle c, e \rangle, \quad |c| = 1, \quad |e| = 2,
\]
\[
\partial e = c - c = 0, \quad \partial c = 1 - e^x - e^p + a^2 e^x e^p
\]
Knot contact homology

The trefoil $T$ (differential in degree 1):

$$\mathbb{C}[e^{\pm x}, e^{\pm p}, Q^{\pm 1}]\langle a_{12}, a_{21}, b_{12}, b_{21}, c_{ij}, e_{ij}\rangle_{i,j\in\{1,2\}},$$

$$|a_{ij}| = 0, |b_{ij}| = |c_{ij}| = 1, |e_{ij}| = 2,$$

\[
\begin{align*}
\partial b_{12} &= e^{-x}a_{12} - a_{21}, \\
\partial b_{21} &= e^{x}a_{21} - a_{12}, \\
\partial c_{11} &= e^{p}e^{x} - e^{x} - (2a^{2} - e^{p})a_{12} - a^{2}a_{12}a_{21}, \\
\partial c_{12} &= a^{2} - e^{p} + e^{p}a_{12} + a^{2}a_{12}a_{21}, \\
\partial c_{21} &= a^{2} - e^{p} + e^{p}e^{x}a_{21} + a^{2}a_{12}a_{21}, \\
\partial c_{22} &= e^{p} - 1 - a^{2}a_{21} + e^{p}a_{12}a_{21},
\end{align*}
\]
Consider $CE(\Lambda_K)$ as a family over algebras over $(\mathbb{C}^*)^3$, where points in $(\mathbb{C}^*)^3$ correspond to values of coefficients $(e^x, e^p, a)$.

**An augmentation** of $CE(\Lambda_K)$ is a chain map

$$\epsilon: CE(\Lambda_K) \to \mathbb{C}, \quad \epsilon \circ \partial = 0,$$

of unital dg-algebras ($\mathbb{C}$ in degree 0, trivial differential).

**The augmentation variety** $V_K$ is the algebraic closure of

$$\{(e^x, e^p, a) \in (\mathbb{C}^*)^3 : CE(\Lambda_K) \text{ has augmentation}\}.$$

It has codim 1 and the generator of the corresponding ideal is called the augmentation polynomial $A_K$. 
The augmentation variety

**Unknot:**

$$A_U(e^x, e^p, a) = 1 - e^x - e^p + a^2 e^x e^p.$$  

**Trefoil:**

$$A_T(e^x, e^p, a) = a^6 - a^6 e^x - a^4 e^p + a^4 e^x e^p$$
$$- 2a^2 e^x e^{2p} + 2a^4 e^x e^{2p} + a^2 e^x e^{3p}$$
$$- e^{2x} e^{3p} - a^2 e^x e^{4p} + e^{2x} e^{4p}.$$
Exact Lagrangian fillings $L$ of $\Lambda_K$ in $T^* S^3$ induce augmentations

$$\epsilon_L(c) = \sum_{|c| = 0} |M_A(c)| e^A.$$ 

Coefficients – induced map on homology.

Exact fillings in $T^* S^3$ ($a = 1$): $L_K$ and $M_K \approx S^3 - K$, $e^p = 1$ and $e^x = 1$ belong to $V_K|_{a=1}$ for any $K$.

For the unknot $A_U(e^x, e^p, a = 1) = (1 - e^x)(1 - e^p)$. 
The non-exact $L_K \subset X$ supports closed holomorphic disks and the exact case map is not a chain map because of new boundary phenomena:

Use Fukaya-Oh-Ohta-Ono bounding chains: $\sigma_u$ for each rigid disk $u$ that connects its boundary in $L_K$ to a multiple of a standard generator at infinity.
**Generalized holomorphic disks:** ordinary holomorphic disks with all possible insertions of $\sigma$’s along the boundary. In $\mathcal{M}_A(c; \sigma)$ boundary bubbling becomes interior points.

$$\epsilon_L(c) = \sum_{|c|=1} \mathcal{M}_A(c; \sigma)e^A$$

is a chain map provided $p = \frac{\partial W_K}{\partial x}$. The substitution counts generalized disks at infinity.
$p = \frac{\partial W_K}{\partial x}$ parameterizes a branch of the augmentation variety if $W_K(e^x)$ is the disk potential counting generalized disks. We can use this to count disks.

**Unknot:**

$$e^p = \frac{1 - e^x}{1 - a^2 e^x}, \quad p(x) = \frac{\partial W_U}{\partial x} = \log(1 - e^x) - \log(1 - a^2 e^x),$$

$$W_U(e^x) = \sum_{d \geq 0} \frac{1}{d^2} (1 - a^{2d}) e^{dx}.$$
Higher genus generalized curves: map from the $U(1)$-skein $(q = a)$ to $q^{\text{writhe}} \times \text{homology}$. Then $e^p e^x = q e^x e^p$.

Eliminating pos and neg punctures gives

$$\hat{A}_K(e^x, e^p, q, a) \nabla_K = 0$$
Large $N$ duality, $\Psi_K$ is the colored HOMFLY. So $\hat{A}_K(e^x, e^p, q)$ is its recursion relation. (At infinity there are no actual higher genus curves and 1-dimensional curves can be found combinatorially.)

For the unknot, only disks. The operator equation is

$$\hat{A}_U(e^x, e^p, a) = (1 - e^x - e^p - a^2 e^x e^p) \Psi_U = 0.$$  

For the topological vertex take $a \to 0$, count generalized curves over only the '1-disk'. We get

$$\psi(e^x) = \sum_{d \geq 0} \frac{e^{dx}}{(1 - q)(1 - q^2) \ldots (1 - q^d)},$$

which agrees with GW-counts for $q = e^{g_s}$. 
Open GW potential and higher genus curves at infinity

**Trefoil**

\[
\begin{align*}
H(b_{12}) &= e^{-x} \partial_{a_{12}} - \partial_{a_{21}} + O(a_{ij}) \\
H(c_{11}) &= e^{x} e^{p} - q^{-1} e^{x} - ((1 + q^{-1}) a^{2} - e^{p}) \partial_{a_{12}} - a^{2} \partial_{a_{12}}^{2} \partial_{a_{21}} + O(a_{ij}) \\
H(c_{21}) &= a^{2} - e^{p} + e^{x} e^{p} \partial_{a_{21}} + a^{2} \partial_{a_{12}} \partial_{a_{21}} + (q^{-1} - 1) e^{x} a_{12} \\
&\quad + (q^{-1} - 1) a^{2} a_{12} \partial_{a_{12}} + O(a_{ij}^{2}) \\
H(c_{22}) &= e^{p} - 1 - a^{2} \partial_{a_{21}} + e^{p} \partial_{a_{12}} \partial_{a_{21}} + (q - 1) a^{2} a_{12} \\
&\quad + (q - 1) e^{p} a_{12} \partial_{a_{12}} + O(a_{ij}^{2}),
\end{align*}
\]

Elimination gives

\[
\hat{A}_{T}(e^{x}, e^{p}, a, q) = qa^{6} e^{3p}(a^{2} - q^{-3} e^{2p})(a^{2} - q^{-1} e^{p}) \\
+ q^{-5/2} (a^{2} - q^{-2} e^{2p}) ((q^{2} e^{2p} + q^{3} e^{2p} - q^{3} e^{p} + q^{4}) a^{4} \\
- (qe^{3p} + q^{3} e^{2p} + q e^{2p}) a^{2} + e^{4p}) e^{x} \\
+ (a^{2} - q^{-1} e^{2p})(e^{p} - q) e^{2x}.
\]
It was observed that the generating function for the colored HOMFLY can be written as a quiver partition function for a symmetric quiver. The geometry behind such expressions can be understood if we assume that there is a finite set of basic holomorphic disks (the quiver nodes) attached to $L_K$ such that all holomorphic curves lie in a neighborhood of $L_K \cup \{\text{basic disks}\}$.
As for generalized disks, we must keep track of the linking number between disks to count generalized curves. The result is an expression of the following form:

$$
\Psi_K(e^x, a, q) = \psi \left( e^{x_1} e^{\sum_{j=1}^n C_{1j} g_s \partial x_j} \right) \ldots \psi \left( e^{x_m} e^{\sum_{j=1}^n C_{mj} g_s \partial x_j} \right)
$$

$$
= \sum_{(d_1, \ldots, d_m) \in \mathbb{Z}_+^m} (-q)^{\sum_{ij} C_{ij} d_i d_j} \prod_{j=1}^m \frac{e^{d_j x_j}}{(q^2, q^2)_{d_j}},
$$

where $e^{x_i} = q^{n_i} a^{k_i} e^{l_i x}.$

**Geometric characters of nodes:** $C_{ij}$ is linking between disks $i$ and $j$, $C_{ii}$ self-linking or framing data for attaching the disk, $n_i$ is 4-chain intersections (invariant self-linking minus framing), $(k_i, l_i)$ homology class in $H_2(X, L_K)$. 
Basic holomorphic disks and quivers

Unknot

\[ u \]

\[ e^x \quad a^2 e^x \]

Trefoil

\[ a^2 q^{-2} e^x \]

\[ a^4 q^{-3} e^x \]

\[ a^6 q^{-4} e^x \]

\[ a^4 q^{-4} e^x \]

\[ a^4 q^{-3} e^x \]

\[ a^4 q^{-1} e^x \]
Non-uniqueness of quivers

Different quivers can give rise to the same partition function. There are two main sources.
Refined quiver partition function

**Refinement**, brief discussion. The refined partition function comes from considerations of $U(1)$-actions and M-theory it is the same as before,

$$
\Psi_K(e^x, a, q, t) = \sum_{(d_1, \ldots, d_m) \in \mathbb{Z}_+^m} (-q)^{\sum_{ij} C_{ij} d_i d_j} \prod_{j=1}^m \frac{e^{d_j x_j}}{(q^2, q^2)_{d_j}},
$$

but where $t$ is a new variable keeping track of how disks are attached to $L_K$:

$$
e^{x_i} = q^{n_i} (-t)^{C_{ii}} a^{k_i} e^{l_i x}.
$$

(Geometrically it can be found by looking at the kernel vector fields of the linearized Cauchy-Riemann operator with a simple pole on the boundary.)
The refined quiver partition function determines the quiver. It also determines a collection of ‘fake quivers’ as follows. The coefficient of $e^{kx}$ in the expansion $\Psi_K(a, x, q, t)$ can be expressed uniquely as the corresponding coefficient for $\Psi_{D_k}(a, x, q)$, where $D_k$ is a quiver with mutually non-linking nodes at level 1.

Conjecturally, the quiver nodes of the fake quiver $D_k$ are the generators of colored HOMFLY homology. To get to $\text{sl}_N$ homology, substitute $a = q^N$ and cancel disks that form canceling pairs ‘from odd to even degree’.