AN ANALYSIS OF A COMBINATORIC GAME

RYAN SUTTON

ABSTRACT. In this report, we discuss the further analysis performed on the Nim like games and applications in both graph theory and combinatorics.

1. Previous Definitions

This section contains a list of definitions used in a previous analysis and explained rigorously. These definitions will be used once again in this progress report.

Definition 1 (Game). A game is a set of subgames available from all possible moves from a left and a right player. A game $G$ is expressed as:

$$G = \{H_L|H_R\}$$

Where $H_L$ and $H_R$ are sets of games.

Definition 2 (Impartiality). A game of the form $\{H_L|H_R\}$ is considered impartial if:

$$H_L \simeq H_R$$

Theorem 1 (Nimlets). An $*n$ game is equal to:

$$\forall n \in \mathbb{N}, *n = \{*0, \ldots, *(n-1)|*0, \ldots, *(n-1)\}$$

$n = 0, *n = \{|\}$.

This follows as for each game with one pile and an $n$ number of objects within this pile, the player can make a move so that there are any number of objects in the pile from $0...(n-1)$

Definition 3 (Negation of a Game). Let $G$ be a game and $G = \{H_L|H_R\}$. Then the negation of $G$ is equal to

$$-G = \{H_R|H_L\}$$

Corollary 2. Let $G$ be an impartial game. Then

$$G = -G$$

And

$$G + G = *0$$
To express games with multiple piles, let \( n \) be the number of piles and \( a_1, \ldots, a_n \) be the number of objects in each pile respectively. Then the game is equal to

\[
\sum_{i=1}^{n} a_i
\]

With this definition, we can extrapolate a few principles of addition.

**Theorem 3** (Symmetry Principle). For all numbers \( n \) in \( \mathbb{Z}^+ \), \( *n + *n = *0 \).

**Proof.** Let \( a \in \mathbb{Z}^+ \). Then \( *a \) is impartial and \( *a = -*a \). Therefore

\[
*a + *a = *a -*a = *0
\]

To add two arbitrary games, we take the binary exclusive or of the two games.

**Theorem 4** (Nimlet Summation Principle). Let \( a, b \in \mathbb{Z}^+ \) and \( a \neq b \). Then

\[
*a + *b = *(a_{base \ 2} \oplus b_{base \ 2})
\]

**Theorem 5** (Sprague-Grundy Theorem). Any finite, impartial, two-person game is equivalent to a Nim game.

**Definition 4** (Finite Impartial Two-Person Game). A finite, impartial, two-person game is defined recursively:

1. \( \{\} \) is a finite, impartial, two person game.
2. If \( S \) is a set of finite, impartial, two person games, then \( S \) is also a finite, impartial, two person game.

**Lemma 6** (Mex Principle). Let \( G = \{T|T\} \) be a finite, impartial, two person game. Then \( G = *mex(\mathcal{T}) \)

2. **Original Die Game Analysis**

The n-sided Die Game has a simple set of rules when restricted to even n’s. The players choose an arbitrary remainder to start the game at, usually a value greater than \( 4n \) so the game is not trivial. The first player moves the die to a number and that number is subtracted from the remainder. The next player selects the next number with the following restrictions:

1. Let \( N_l \) be the last number selected.
2. Let \( R_{current} \) be the current remainder.
3. Any number \( N_c \) may be chosen from \( \{1, \ldots, n\} \) such that:
   a. \( N_c \neq N_l \)
   b. \( N_c \neq n + 1 - N_l \)
   c. \( R_{current} - N_c \geq 0 \)
The game ends when a player cannot make a move from the current position. In this case, the opposing player wins.

Since the n-sided Die game is a finite, impartial, two person game, it can be assigned Nimlets by the Sprague-Grundy Theorem and specific Nimlets by the Mex Principle. After an analysis of several differing values of n, we can classify the Nimlets of the n-sided die game algorithmically. Primarily, we have two methods of classifying the games:

(1) Periodicity of the Nimlets.
(2) Maximum Nimlet generated during the analysis.

A list of the periodicity of several even n-sided die games as well as the maximum Nimlets seen are listed below:

<table>
<thead>
<tr>
<th>n</th>
<th>Periodicity</th>
<th>Nimlets</th>
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<tbody>
<tr>
<td>2</td>
<td>N/A</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>N/A</td>
<td>2</td>
</tr>
<tr>
<td>6</td>
<td>9</td>
<td>4</td>
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</table>

Both the periodicity and the Nimlets have an interesting correlation that is not immediately apparent. One conjecture can be made towards the maximum nimlet seen in one game. First let us define a principle involving the maximum possible Nimlet based on the number of positions in a finite, impartial, two person game.

**Theorem 7** (Maximum Reach Principle). *For any game $G$ whose number of positions $|G| = n$, then $G$ can be at most $\ast n$*

**Proof.** Let $n$ be a positive integer and let $G$ be a game with $|G| = n$. Then by the Mex Principle, the highest Nimlet associated with $|G|$ will occur when the positions within $G$ are the games $\{\ast 0, \ast 1, \ldots, \ast (n - 1)| \ast 0, \ast 1, \ldots, \ast (n - 1)\}$. Then at most $G \simeq \ast n$ for all values of $n \geq 0$. \qed

**Lemma 8.** *For any $n \geq 2$, the n-sided die game will not contain a position whose Nimlet is greater than $\ast (n - 2)$.*

**Proof.** Let $n$ be a natural number. Let $M_{ij}$ be the last position played from 1 to $n$. Then there is a maximum of $n-2$ moves resulting from the restrictions of the n-sided die game. Let $A$ be the current game resulting from the current available positions. Then $|A| = n - 2$ from the previous analysis. Therefore, by the Maximum Reach
Principle, the maximum Nimlet associated with \( A \) must be \(* (n - 2)\). Therefore, there cannot exist a Nimlet greater than \(* (n - 2)\) in the n-sided die game.

\[\square\]

3. Filtered Graphs

From the last analysis, we discovered that there was a limitation to a simple patterned analysis. Even when attempting to a recursive method of analysis, the pattern was not apparent until using 5th order analysis, at which point the simplification of the equation was impossible. We then moved on to an analysis using graph theory. We first defined a new type of graph we called filtered graphs:

**Definition 5 (Filtered Graph).** Let \( V \) be a vertex space representing positions in a game. Let \( E \) be an edge map which represents all possible moves from one position to another, expressed through \( E(V_1, V_2) \) to move from \( V_1 \rightarrow V_2 \). Let \( T \) be a function such that:

\[
T : V \rightarrow \mathcal{P}(V) \quad v \in V, T(v) = \{ v_2 \in V | \exists E(v_1, v_2) \in E \}
\]

Let \( F(V, E) \) be a graph with vertex space \( V \) and edge map \( E \). Then if there does not exist a \( n \) such that \( v \in T^n(v) \) for any \( v \in V \), then \( F(V, E) \) is a filtered graph.

These graphs are unique in the fact that they model a finite, impartial, two person game. Specifically the graph presents a recursive pattern when determining if a position is a P or an N position.

**Definition 6 (P/N positions).** Let \( v \in V \) then:

\[
PN(v) = \begin{cases} 
1 : \exists v_1 \in T(v), \quad PN(v_1) = 0 \\
0 : \forall v_1 \in T(v), \quad PN(v_1) \neq 0
\end{cases}
\]

This is known as the standard position map.

We shall define the isomorphism for the filtered graphs.

**Definition 7 (Filtered Graph Isomorphism).** Let \( F_1(V_1, E_1, PN_1), F_2(V_2, E_2, PN_2) \) be two filtered graphs. Then \( \gamma \) is an isomorphism from \( F_1 \rightarrow F_2 \) if and only if:

1. \( \gamma \) provides a bijection between \( V_1 \) and \( V_2 \).
2. \( \gamma \) provides a surjection between \( E_1 \) and \( E_2 \).
3. The resulting PN values of \( F_2 \) are equivalent to \( F_1 \) for all vertices.

While this definition is correct, it does not produce better results than our previous definitions in the patterned based analysis. Our focus this semester was to attempt to find a function \( \Gamma \) which would simplify a graph to a series of Nim-like graphs which would allow us to determine the PN value of a specified position without knowledge of the entire graph. However, we do not know if there exists only one unique graph that the original graph can be reduced to. While the Sprague-Grundy theorem seems to imply this, there may be many different functions which reduce the graph but destroy information from positions not being analyzed.
4. Conclusions

Currently, we have reached a point where we believe that the reduction problem may have an NP-Hard complexity. While this does not prevent additional study and work, it may be more fruitful to focus upon a simpler or more complete problem. We had discovered that a position for any finite, impartial, two person game can be modeled using graph theory which may provide a method of further algorithm analysis and may provide a method for decreasing the time requirement to NP-complete. We hope to continue this research in the future.

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Ryan Sutton, Department of Mathematics, 138 Cardwell Hall, Kansas State University, Manhattan, KS 66506, USA.
E-mail address: rs4090@k-state.edu