BACKWARD SHIFT REALIZATION OF DISCRETE ANALYTIC FUNCTION

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1. Introduction

A function \( f : D_r \subseteq \mathbb{C} \rightarrow \mathbb{C} \) where \( D_r \) is an open disk with center at \( a \) of radius \( r \), is said to be analytic in \( D_r \) if either one of the following equivalent conditions is satisfied:

(i) For every \( z \in D_r \), \( f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h} \), where \( h \in \mathbb{C} \), exists.

(ii) In \( D_r \), \( f \) admits the Taylor expansion \( f(z) = \sum_{n=0}^{\infty} C_n(z-a)^n \); that is: \( \sum_{n=0}^{\infty} C_n(z-a)^n \) converges to \( f(z) \) in \( D_r \). In particular \( f(z) = \sum_{n=0}^{\infty} C_n(z-a)^n \) is uniform convergent on every compact subset of \( D_r \), therefore \( f(z) \in \mathcal{C}^\infty \).

(iii) For every smooth \( \gamma : [a, b] \rightarrow D_r \) such that \( \gamma(a) = \gamma(b) \), \( \oint_{\gamma} f(z)dz := \int_{a}^{b} f(\gamma(t))\gamma'(t)dt = 0 \).

Now, given a function \( f \) defined on the integer lattice in the complex plane (i.e. \( \Lambda = \mathbb{Z} \times i\mathbb{Z} \)), say \( f : \Lambda \rightarrow \mathbb{C} \), the question arises: When is \( f \) analytic or, as we shall say, discrete analytic? Since \( f \) is defined on \( \mathbb{Z}^2 \), \( |h - 0| \geq 1 \); therefore condition (i) cannot be used because the notion of limit is not really feasible. Also note that the disk \( D_r \) contains only finitely many integer points, so uniform convergence is meaningless, therefore condition (ii) is not helpful either. Condition (iii), on the other hand, can be approximated for discrete function as:

For every \( \{z_1, ..., z_n, z_{n+1} = z_1\} \) such that \( |z_{k+1} - z_k| = 1 \)

\[ \oint_{\gamma} f(z)dz \approx \sum_{k=1}^{n} \frac{f(z_{k+1}) + f(z_k)}{2}(z_{k+1} - z_k) = 0 \]

Therefore, this condition can be used to define discrete analytic functions. Now note that the interior of every simple closed loop in \( \Lambda \) can be divided into unit squares; therefore:

\[ \sum_{k=1}^{n} \frac{f(z_{k+1}) + f(z_k)}{2}(z_{k+1} - z_k) = 0 \iff \frac{f(z + 1 + i) - f(z)}{1 + i} = \frac{f(z + 1) - f(z + i)}{1 - i} \]

This conditions gives the discrete version of the Cauchy-Riemann equations

(1.1) \( (1 - i)(\delta_x f)(z) + (i + 1)(\delta_y)(z) + (\delta_x \delta_y f)(z) = 0 \)

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In fact, a function \( f : \Lambda \to \mathbb{C} \) is said to be discrete analytic if it satisfies 1.1. This notion of discrete analyticity was introduced by J. Ferrand in [2].

As a result of this definition, if \( f \) is discrete analytic its restriction to the even and odd sublattices are discrete harmonic. That is: \( f : \Lambda_e = \{(x, y) \in \Lambda : x + y = 2k \ k \in \mathbb{Z}\} \to \mathbb{C} \) and \( f : \Lambda_o = \{(x, y) \in \Lambda : x + y = 2k + 1 \ k \in \mathbb{Z}\} \to \mathbb{C} \) are discrete harmonic, meaning that the value of \( f \) at \((x, y)\) is the average its value at the neighboring points.

In further results it was shown by C. Mercat in [3] that this definition of discrete analyticity generalizes to a large class of planar graphs.

Naturally we ask the question: are sums, products and quotients, of discrete analytic functions, discrete analytic themselves? Turns out that sums of discrete analytic functions are discrete analytic but products, and for instance quotients, need not to be discrete analytic. For example it can be checked, that \( f(z) = z \) and \( f(z) = z^2 \) are discrete analytic but \( f(z) = z^3 \) is not. We, therefore, ask the question of how to construct discrete analytic polynomials of arbitrary degree. D. Alpay, P. Jorgensen, R. Seager and D. Volok addressed this question in [1] where they introduced a new product operator \( Z \) which preserves discrete analyticity. This product operator \( Z \) induces a product of discrete analytic functions, named by the authors as Cauchy-Kovalevskaya product and defined as

\[
(\delta^n f)(x) = f(x+1) - f(x), \hspace{1cm} (\delta^n f)(0) = \frac{(\delta^n f)(0)}{n!} x^{[n]}
\]

where \( f(x, 0) = c_0 + c_1 x + ... + c_n x^n \) and \( (\delta g)(x, y) = xg(x, y) + iyg(x, y+1) + g(x, y-1) \). This product has the desired property of preserving discrete analyticity; and the additional property that on the \( x \)-axis this product is equal to the point-wise product; that is: \( (f \odot g)(x, 0) = (g \odot f)(x, 0) = f(x, 0)g(x, 0) \). This way we can construct discrete analytic polynomials of arbitrary degree. It was shown in the same paper that if \( p \) is a discrete polynomial defined in \( \mathbb{Z} \) there are infinitely many discrete analytic extension of \( p \) to the integer lattice \( \Lambda \) but only one of them is itself a polynomial in two variables.

For a discrete function \( f : \mathbb{Z} \to \mathbb{R} \) it is well known that its power series expansion (the discrete analog of the Taylor expansion) is given by

\[
f(x) = \sum_{n=0}^{\infty} \frac{(\delta^n f)(0)}{n!} x^{[n]} \text{ where } (\delta f)(x) = f(x+1) - f(x), \hspace{1cm} x^{[n]} = \prod_{j=0}^{n-1} (x - j)
\]

If we regard \( x^{[n]} \) as a discrete polynomial, then we know that it extends to a unique discrete analytic polynomial in two variables which we will call \( \zeta_n \). It turns out that \( \forall n \in \mathbb{Z}_+ \)

\[
\limsup_{n \to \infty} |n!\zeta_n|^{\frac{1}{n}} = \frac{1}{\sqrt{2}}
\]

Based on that results, In the latter paper, the authors introduced the notion of discrete analytic expandable function as a function \( f : \Lambda_+ = \mathbb{Z}_+ \times \mathbb{Z} \to \mathbb{C} \) such that:
if, there is an $n$ in $D$ the notion of expandability allows to extend discrete functions from $\mathbb{Z}$ to introduce the backward shifts of analytic functions. Let $f$ be analytic on $\Lambda$ and discuss the implications of this extension. Once a function is expandable it is known to be expandable in the whole plane, it is a fair question to ask how are the values on the real axis related to the values on the imaginary axis; that is, how is $f(x, 0)$ related to $f(0, y)$. This question is addressed in section 2.2. The answer of this question also motivates a new, more natural, definition of expandability.

In this paper we extend the notion of expandability in $\Lambda_+$ to expandability in the whole plane (i.e. $\Lambda$) and discuss the implications of this extension. Once a function is known to be expandable in the whole plane, it is a fair question to ask how are the values on the real axis related to the values on the imaginary axis; that is, how is $f(x, 0)$ related to $f(0, y)$. This question is addressed in section 2.2. The answer of this question also motivates a new, more natural, definition of expandability.

In the continuous case, when solving interpolation problems it is particularly useful to introduce the backward shifts of analytic functions. Let $f : D \subseteq \mathbb{C} \to \mathbb{C}$ be analytic in $D$, then $f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n = c_0 + c_1 (z-a) + c_2 (z-a)^2 + \ldots$ for some $a \in D$. Now consider the backward shift of $f$ centered at zero, $(R_0 f)(z) = c_1 + c_2 z + c_3 z^2 + \ldots$, here we see that $(R_0 f)(z) = \frac{f(z) - f(0)}{z}$. It can be shown that $f$ is rational if, and only if, there is an $n \in \mathbb{Z}_+$ such that $\forall m > n (R_0^m f)(z) = \alpha_1 (R_0 f)(z) + \alpha_2 (R_0^2 f)(z) + \ldots + \alpha_n (R_0^n f)(z)$; this ensures that $f$ can be realized as $f(z) = f(0) + zC(I-zA)^{-1}B$ where $A$, $B$, and $C$ are matrices of suitable dimensions. This realization of rational functions is well understood, and particularly useful in the linear input-state-output system theory in electrical engineering. In section ?? we introduce backward shifts of discrete analytic functions, discuss the relation with discrete analytic rational functions and establish conditions under which these rational functions can be “realize” in terms of their backward shifts.

2. Discrete Analytic Expandable Functions

2.1. Expandability on the upper half plane. As mentioned in the introduction, a discrete analytic function $f : \Lambda_+ \to \mathbb{C}$ is said to be expandable if it can be written as power series in terms of discrete polynomials $\zeta_n(z)$, where $\zeta_n(z)$ is discrete analytic on $\Lambda_+$ for all $n$, and $\zeta_n(x+0i) = x[n]$.

To extend this definition to more general “half planes”, say $\Lambda_{x \geq x_0} = \{(x, y) \in \Lambda : x \geq x_0\}$ we set $f : \Lambda_{x \geq x_0} \to \mathbb{C}$ and consider the shift $f(x+x_0, y)$ defined in $\Lambda_+$. We check if this shifted function is expandable on $\Lambda_+$; that is: we check if $f(x+x_0, y)$ satisfies

\begin{align*}
\text{(i) } f(x+x_0, 0) &= \sum_{n=0}^{\infty} C_n x[n], \text{ where } C_n = \frac{\delta^n[f(x+x_0, 0)]}{n!} \bigg|_{x=0} = \frac{\delta^n[f_0]}{n!} \bigg|_{x=x_0}, \\
\text{and } \limsup_{n \to \infty} |C_n n!|^{1/n} &< \sqrt{2}. \\
\text{(ii) } f(x+x_0, y) &= \sum_{n=0}^{\infty} C_n \zeta_n(x, y).
\end{align*}
If this is the case, we say that $f$ is expandable in $\mathbb{Z}_{x \geq x_0} \times \mathbb{Z}$.

This definition allows us to assume, without lost of generality, that the domain of expandability is always $\Lambda_+$, however such definition restricts expandability to half planes and is, itself, not a very natural definition. In section 2.2 we will give a more natural definition that will follow from the discussion preceding it.

It is worth mentioning that the discrete analytic polynomials $\zeta_n$ have generating function

$$E(t, z) = \sum_{n=0}^{\infty} \frac{t^n \zeta_n(z)}{n!} = (1 + t)^x \left( \frac{1 - i + t}{1 + i + t} \right)^y$$

(We refer the reader to [1] for details on expandability and discrete analytic polynomials $\zeta_n$.) If a discrete function $f : \mathbb{Z}_+ \rightarrow \mathbb{C}$ is expanded in such way, we shall abuse notation denoting its expansion by $f(Z)$. This allows us to expand discrete functions to discrete analytic functions in the “right half plane”. Here, we extend this discrete analytic expansion to the “upper half plane” and explain some of the consequences of this extension.

Let $f$ be a discrete function defined on $\mathbb{Z}_+$, we know it is uniquely expanded to a discrete analytic function on the right half plane $f(z) = \sum_{n=0}^{\infty} a_n \zeta_n(z)$. Loosely speaking, this expansion can then be restricted to the imaginary axis and expanded form there to a discrete analytic function on the upper half plane. However there is an important question of convergence of this “new” expansion and wether or not this expansion agrees with the original expansion in the quadrant where they are both defined. Consider the generating function of the basis $\{\zeta_n\}$:

$$E(t, z) = \sum_{n=0}^{\infty} \frac{t^n \zeta_n(z)}{n!} = (1 + t)^x \left( \frac{1 - i + t}{1 + i + t} \right)^y$$

It is not hard to see that, when restricted to the real axis, the series converges for $|t| < \sqrt{2}$. Recall that

$$E(t, x) = \sum_{n=0}^{\infty} \frac{t^n \zeta_n(x)}{n!} = \sum_{n=0}^{\infty} \frac{t^n x[n]}{n!}$$

Now consider the restriction of $E$ to the imaginary axis: $E(t, iy) = \sum_{n=0}^{\infty} c_n y[n]$ where $c_n$ is the $n^{th}$ Taylor coefficient of $E$, i.e. $c_n = \left( \frac{\delta^n_y E(t, iy)}{n!} \right)_{|y=0}$. We recall that $(\delta_y E)(t, iy) = E(t, i(y + 1)) - E(t, iy)$ and since $E(t, iy) = \left( \frac{1 - i + t}{1 + i + t} \right)^y$ we get:

$$(\delta_y E)(t, iy) = \left( \frac{1 - i + t}{1 + i + t} \right)^{y+1} - \left( \frac{1 - i + t}{1 + i + t} \right)^y = E(t, iy) \left[ \frac{1 - i + t}{1 + i + t} - 1 \right] = E(t, iy) \left[ \frac{t(i - 1)}{1 + i + t} \right]$$

From here we see:

$$(\delta^n_y E)(t, iy) = E(t, iy) \left[ \frac{t(i - 1)}{1 + i + t} \right]^n$$
But since $E(t, 0) = 1$ we get:

$$(\delta_y^n E)(t, 0) = \left[ \frac{t(i - 1)}{1 + i + t} \right]^n c_n n!$$

In view of the definition of expandability, in order for the expansion from the imaginary axis to converge, we need $|c_n n!|^{1/n} < \sqrt{2}$ which would require $|\frac{t(i - 1)}{1 + i + t}| < \sqrt{2}$. This tells us that the series will converge for $t \in \mathbb{C}$ such that $re(t) + im(t) > -1$. This implies that, even though the expansion of $E(t, z)$ was originally convergent for any $t$ such that $|t| < \sqrt{2}$, extending the expansion to the second quadrant restricts the convergence to a smaller disk, i.e. the disk $\{t : |t| < \frac{1}{\sqrt{2}}\}$.

2.2. **Restriction to the axes.** Once a Discrete analytic function is defined, by means of its expansion, on the right and upper half planes we want to know if there is a relation between the values of the function on the positive real axis and the values on the positive imaginary axis. Let $f(z) : \sum_{n=0}^{\infty} c_n \zeta_n(z)$ be a discrete analytic expandable function, to study the relation between the values of $f$ on the axes we consider once again the generating function of the discrete analytic polynomial $\zeta_n$:

$$E(t, z) = \sum_{n=0}^{\infty} \frac{t^n \zeta_n(z)}{n!}.$$  

We know $(\delta_x \zeta_n)(z) = n \zeta_{n-1}(z)$ from where we can see that $\delta_x E = tE$. As a function of $z$, $E(t, z)$ is discrete analytic, therefore by Cauchy-Riemann equation $(1 - i)\delta_x E + (i + 1)\delta_y E + \delta_x \delta_y E = 0$ we get

$$\delta_y E = -\frac{(1 - i)t}{1 + i + t}$$

which justifies the equality

$$E(y, z) = (1 + t)x \left(1 - \frac{(1 - i)t}{1 + i + t}\right)^y.$$  

Now, if we regard $E$ as a function of $t$ we see that, by Taylor expansion,

$$\zeta_n(z) = \frac{d^n}{dt^n} \left[ (1 + t)^x \left(1 - \frac{(1 - i)t}{1 + i + t}\right)^y \right]_{t=0}.$$  

We now let $F(t, z) = \sum_{n=0}^{\infty} \frac{t^n \sigma_n(z)}{n!}$ where $\sigma_n$ is a discrete analytic polynomial such that $\sigma_n(0 + iy) = y^n$; that is: $\sigma_n$ is the discrete analytic expansion of the discrete polynomial $y^n$. As before, when regarded as a function of the discrete variable $z$, $F(t, z)$ is discrete analytic. Therefore, by Cauchy-Riemann equations and since $\delta_y \sigma_n = n \sigma_{n-1}$ (i.e. $\delta_y F = tF$) we get

$$\delta_x F = -\frac{(1 + i)t}{1 - i + t} F,$$  

hence: $F(t, z) = (1 + t)^y \left(1 - \frac{(1 + i)t}{1 - i + t}\right)^x$. 

It can be verified from here that the transformations \( T : t \mapsto \frac{(1 - i)t}{1 + i + t} \) and \( S : s \mapsto \frac{(1 + i)s}{1 - i + s} \) are inverses of each other and therefore \( E(t, z) = F\left(\frac{-(1 - i)t}{1 + i + t}, z\right) \). This suggests that, as in the continuous case, there is a relation between the values of a discrete analytic expandable function on the real axis and its values on the imaginary axis.

To describe this relation we want to find a transformation \( A : \sum_{n=0}^{\infty} \frac{t^n}{n!} x^{[n]} \mapsto \sum_{n=0}^{\infty} \frac{T(t)^n}{n!} x^{[n]} \) such that, together with discrete analyticity, guarantees expandability. To begin, regard \( H^2 = \{ \sum_{n=0}^{\infty} c_n z^n : \sum_{n=0}^{\infty} |c_n|^2 < \infty \} \) as a Hilbert space with \( \langle \sum_{n=0}^{\infty} c_n z^n, \sum_{n=0}^{\infty} d_n z^n \rangle = \sum_{n=0}^{\infty} c_n d_n \) and \( || \sum_{n=0}^{\infty} c_n z^n || = (\sum_{n=0}^{\infty} |c_n|^2)^{1/2} \). It can be checked that \( H^2 \) is a reproducing Kernel Hilbert Space; that is \( \forall g \in H^2 \) and \( w \in \mathbb{D} = \{ z \in \mathbb{C} \mid |z| < 1 \} \), \( \exists k \in H^2 \) such that

\[
k_w(z) = \frac{1}{1 - zw} = \sum_{n=0}^{\infty} z^n w^n \in H^2
\]

Moreover if \( g(z) = \sum_{n=0}^{\infty} c_n z^n \),

\[
\langle g(z), k_w(z) \rangle = \sum_{n=0}^{\infty} c_n w^n = g(w).
\]

In particular

\[
\langle g, k_w(z) \rangle = \frac{1}{2\pi i} \oint_{|z|=1} \frac{g(z)}{1 - zw} \frac{dz}{z} = \frac{1}{2\pi i} \oint_{|z|=1} \frac{g(z)}{z - w} dz.
\]

Now let \( f \) be a discrete function \( f(x) = \sum_{n=0}^{\infty} c_n x^{[n]} \) where \( \limsup_{n \to \infty} (|c_n|^{1/2} < \sqrt{2} \) and consider its z-transform \( \hat{f}(z) = \sum_{n=0}^{\infty} c_n z^n \) which is analytic for all \( z \) such that \( |z| \leq \frac{1}{\sqrt{2}} \). The transformation \( A \) must satisfy \( \hat{A} : \sum_{n=0}^{\infty} t^n z^n \mapsto \sum_{n=0}^{\infty} T(t)^n z^n \), i.e.

\[
\frac{1}{1 - tz} \mapsto \frac{1}{1 - T(t)z}.
\]

That is: \( k_t \mapsto k_{T(t)} \) so, replacing \( t \) with \( \bar{t} \) and letting \( T^\#(t) = \bar{T}(\bar{t}) \), we get \( k_t \mapsto k_{T^\#(t)} \).

In \( H^2 \) we have \( \langle \hat{A}\hat{f}(z) \rangle = \langle \hat{\bar{f}} k_z, k_z \rangle = \langle \bar{f}, \hat{A}^* k_z \rangle \); but \( \langle \hat{A}^* k_z, k_t \rangle = \langle k_z, \hat{A}^* k_t \rangle = \langle k_z, k_{\hat{T}^\#(t)} \rangle = \frac{1}{1 - T^\#(t) \bar{z}} \).

On the other hand:

\[
\langle \bar{f}, \hat{A}^* k_z \rangle = \langle \bar{f}, k_z (T^\#) \rangle = \frac{1}{2\pi i} \oint_{|z|=1} \frac{\hat{f}(s)}{1 - T(\bar{z})z} \frac{ds}{s}
\]
From the definition of $T$ we can get

$$\langle \hat{f}, \hat{A}^* k_z \rangle = \frac{1}{2\pi i} \oint_{|s| = 1} \frac{\hat{f}(s)(1 + (1 + i)s)}{1 + (1 + i)s + (1 - i)z} ds$$

It is not hard to see that for functions analytic on a neighborhood of zero there are two poles and we get

$$\frac{1}{2\pi i} \oint_{|s| = 1} \frac{\hat{f}(s)(1 + (1 + i)s)}{1 + (1 + i)s + (1 - i)z} ds = \frac{\hat{f}(0)}{1 + (1 - i)z} + \frac{\hat{f}(iz - \frac{1-i}{2})(1 + (1 + i)(iz - \frac{1-i}{2}))}{(1 + i)(iz - \frac{1-i}{2})}$$

From where we can conclude that a discrete analytic function whose values on the axes are related by this transformation (i.e. $\hat{f}(x) = \hat{A}\hat{f}(iy)$) is also expandable.

### 3. Discrete Analytic Rational Functions

For now we consider function of a continuous complex variable. Let $f(z) = \sum_{n=0}^{\infty} c_n z^n$ be the Taylor expansion of a function analytic in a neighborhood of 0. In terms of $c_n$, how to tell whether $f(z)$ is a rational function?

A real number $0.c_1c_2c_3\ldots$ is rational iff $\exists N \in \mathbb{N}$ such that the sequence $c_N, c_{N+1}, c_{N+2}, \ldots$ is periodic.

Similarly, it is known that $f(z)$ is rational iff $\exists N \in \mathbb{N}$ such that the Taylor coefficients $c_n$, starting from $N$, satisfy a finite recurrence relation with constant coefficients.

Formally, for the function $f$ analytic on a neighborhood of zero we define the backward shift centered at zero as

$$R_0 f(z) = \frac{f(z) - f(0)}{z}$$

The name “backward shift” comes from the effect of $R_0$ on the Taylor coefficient of $f(z)$

$$R_0 f(z) = \sum_{n=0}^{\infty} c_{n+1} z^n = c_1 + c_2 z + c_3 z^2 + \ldots$$

With that definition, we have the three following equivalent definitions of rationality:

i. $f(z)$ is rational if $f(z) = p(z)/q(z)$

ii. $f(z)$ is rational if the functions $R_0^n f, n = 0, 1, 2, \ldots$ span a finite-dimensional space.

iii. $f(z)$ is rational if $f(z) = D + zC(I - zA)^{-1}B$

Now, if we intend to define discrete analytic rational functions, we have to define them as rational with respect to the Cauchy-Kovaleskaya product and since this product is extended from the real axis we give the natural definition:
Definition 3.1. An expandable DA function $f(x, y)$ is rational if $f(x, 0)$ is rational in the usual sense: $f(x, 0) = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials.

With that motivation we want to know if we can identify DA rational functions by looking at their coefficients in the expansion $\sum_{n=0}^{\infty} c_n \zeta^n$.

Definition 3.2. For a DA expandable function $f : \Lambda_+ \rightarrow \mathbb{C}$, we define its backward shift as

$$Q_0 f(z) = \sum_{n=0}^{\infty} \frac{c_{n+1}}{n+1} \zeta^n.$$ 

Then we get the following:

**Theorem 1.** Let $f(z)$ be a discrete analytic function, expandable in $\Lambda_+$. Let $\mu$ be the operator $\mu f(z) = f(z + 1)$. Then $\exists \ x_0$ such that $f(z)$ is rational in $\Lambda_{x \geq x_0} = \{z \in \Lambda : \text{Re}(z) \geq x_0\}$ if, and only if, $\exists \ m \in \mathbb{N}$ such that $Q_0 \mu^{m+1} f(z)$ is in the span of the set

$$\{\mu Q_0 \mu^m f(z), \mu^2 Q_0 \mu^{m-1} f(z), ..., \mu^{m+1} Q_0 f(z)\}$$

**Corollary 2.** Let $f$ be as in the previous theorem. Then $\exists \ m \in \mathbb{N}$ such that $\text{DIM SPAN}\{Q_0 \mu^{m+1}, \mu Q_0 \mu^m f(z), \mu^2 Q_0 \mu^{m-1} f(z), ..., \mu^{m+1} Q_0 f(z)\} \leq m + 1$ if, and only if, $f(z) = D + zC \odot [I - zA]^{-\odot} B$ where $A, B, C, D$ are matrices of suitable dimensions.

**References**


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