Proof of Wigner Semicircle Law

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Abstract

Wigner semicircle law is the fundamental result in the random matrix theory. It is a very general concept dealing with the distribution of spectra of high-dimensional symmetric matrices. In fact, matrices satisfying the Wigner semicircle law can be pretty general and the claim is thus of a great importance, maybe even of that of the central limit theorem. The purpose of the research was to completely understand the proof of this famous theorem and subsequently, elaborate the self-contained and rigorously written work summarizing the proof and all the necessary material related.

Introduction

As mentioned, the major goal of the research project was providing the results about one of the most fundamental theorems in the random matrix theory in the most comprehensive, rigorous and clear way. Particularly, the research dealt with the proof of the Wigner semicircle law by the moment method. This is the actually the most famous method mainly because it was first used by Wigner to accomplish the proof. Despite of the popularity and significance of the theorem, the sources dealing with its proof mostly miss one of the three listed characteristics. It is either too brief or not rigorous enough or on the other hand, it is often explained in somewhat vague way, not containing all the auxiliary propositions and helpful derivations. Actually, despite the fact that the claim of the theorem is quite straightforward, it requires a great amount of effort as it is usual in the random matrix theory. Namely, theorems and techniques from different areas of mathematics are necessary to be applied. Some of them are standard procedures, but some represent entirely new concepts. The proof of the Wigner semicircle law is specifically the mixture of probabilistic, combinatorial, analytical and algebraic approaches. Indeed, it is not really easy to satisfy all the three named features in such kind of proof. However, the work resulting from the research tries to accomplish that task and goes step by step through all the important derivations, calculations and explanations. It is thus supposed to be a self-contained text intended for those who want to take a glimpse to the realm of random matrices by understanding the basics of its most famous and fundamental theorem.

To be a more specific about the claim of the semicircle law, it states that the distribution of the mixture of the eigenvalues of a hermitian random matrix converges in law to a semicircle distribution as the dimension of the matrix goes to infinity. In other words, the distribution of the mixture of corresponding eigenvalues can be approximated by a semicircle distribution for high-dimensional matrices. As a matter of fact, the approximation mostly holds even for low-dimensional case. Such facts can be used in the theory of so called unfolding to accurately estimate the associated distribution function based on the strong theoretical result rather than using some usual statistical or numerical approach.

Since the claim is satisfied by numerous kinds of random matrices, the theorem is thus pretty general and shows the power and universality of eigenvalues of large random matrices. Even thought the theorem describes one of the basic properties of random eigenvalues, its proof is on the other hand quite long and demanding. Some advanced results from different mathematical areas need to be applied. In the research, all those necessary techniques were studied to correctly understand and consequently accomplish the most famous of the theorem proofs also known as the moment method.

The last mention of the introductory talk is about the sources which the results were
obtained from. Most of the results provided in this text are actually inspired by the book *Topics in Random Matrix Theory* by Terence Tao. It is the book number [1] in the bibliography at the end of this report.

**Methods and Results**

Actually, there are two versions of the Wigner semicircle law. Particularly, the first one is often denoted as the weak version and the second one as the strong version of the Wigner semicircle law. To prove both of them, various techniques and preparatory propositions had to be studied. Specifically, the basic propositions such as the Chebyshev’s inequality and Borel-Cantelli lemma and the crucial advanced ones such as Talagrand inequality and Carleman continuity theorem were dealt with. Based on them and many other minor claims and concepts related, the moment method proof was investigated carefully and rigorously.

Before proceeding in the description of the proof, it is necessary to provide the claim of the Wigner semicircle law. That of the weak version is presented below.

**Theorem 1.** Let \( W_n = (W_{ij})_{i,j=1}^n \) be a hermitian random matrix with identically distributed elements such that \( E|W_{ij}|^k < +\infty, \forall k \in \mathbb{N} \) and \( E|W_{ij}|^2 = \sigma^2 > 0 \) for \( i \leq j \). Let \( \mu_n \) be the distribution mixture of the matrix eigenvalues. Then

\[
\mu_n(S) \xrightarrow{L} \int_S \frac{I_{(-2\sigma,2\sigma)}}{2\pi \sigma^2} \sqrt{4\sigma^2 - \lambda^2} \, d\lambda
\]

for \( S \in \mathcal{B}(\mathbb{R}) \) as \( n \to +\infty \).

To prove such a claim, Carleman continuity theorem needs to be applied to convert it to an equivalent statement. Specifically, the convergence in distribution is transformed into the convergence of all associated moments. Mathematically written,

\[ E(\Lambda_n)^k \to E(\Lambda)^k, \quad \forall k \in \mathbb{N} \]

as \( n \) goes to infinity. Random variable \( \Lambda_n \) is associated to distribution mixture of eigenvalues and \( \Lambda \) corresponds to the semicircle distribution. That is why the term moment method is used. Benefiting from the properties of hermitian matrices, the expectations above can be expressed as

\[ E(\Lambda_n)^k = \frac{E(\text{Tr}(W_n)^k)}{n^{k/2+1}}. \tag{1} \]

In other words, the behavior of trace of all the matrix powers has to be found for the matrix dimension increasing to infinity. Using the properties of trace, expected value and matrix multiplication, such a quantity can be rewritten as a huge sum

\[ E(\text{Tr}(W_n)^k) = \sum_{1 \leq i_1, \ldots, i_k \leq n} E(W_{i_1 i_2} \cdots W_{i_k i_1}) \]

where the individual summands form an expected value of various matrix elements multiplied by each other. At first sight, it looks like the reformulation did not help much. In fact, it is possible to evaluate the sum using an appropriate representation of its summands. It is particularly the cycle of length \( k \) in the graph with \( n \) vertices and all the possible edges and loops. The value of summand is then determined by the type of the associated cycle.
with respect to the multiplicities of its edges and loops. The value of the whole sum is determined by the number of most numerous cycles yielding non-zero summands. For \( k \) odd, such cycles span \( n^{(k-1)/2} \) vertices and in the even case, there are \( n^{k/2+1} \) vertices spanned. Consequently, the limits of (1) can be expressed as

\[
\lim_{n \to +\infty} E(\Lambda_n)^{2l} = C_l \sigma^2, \\
\lim_{n \to +\infty} E(\Lambda_n)^{2l+1} = 0
\]

for all natural \( l \). The \( C_l \) for \( k = 2l \) where \( l \in \mathbb{N} \) denotes the number of the cycles of length \( k \) which span \( k/2 + 1 \) vertices. Such cycles are called non-crossing cycles and the evaluation of their number is the crucial part of the proof of the weak version of the Wigner semicircle law. To accomplish the evaluation, so called Dyck paths are used as an equivalent representation of those cycles. The numbers \( C_l \) for \( l \in \mathbb{N} \) are then derived to have the form

\[
C_l = \frac{1}{l+1} \binom{2l}{l}
\]

which are the famous Catalan numbers. Because the limits (2) are the same as the moments of the semicircle distribution, the weak version of the Wigner semicircle law is completed according to the Carleman continuity theorem.

The proof of the strong version uses some of the results obtained in the proof of the weak version. Nevertheless, the proceeding is quite different since instead of the convergence in distribution, it is dealt with the almost sure convergence in distribution. The proposition is provided below.

**Theorem 2.** Let \( W_n = (W_{ij})_{i,j=1}^n \) be a hermitian random matrix with identically distributed elements such that \( E|W_{ij}|^k < +\infty, \forall k \in \mathbb{N} \) and \( E|W_{ij}|^2 = \sigma^2 > 0 \) for \( i \leq j \). For \( S \in \mathcal{B}(\mathbb{R}) \) define \( N_n(S) = \# \{ i \in \hat{n} \mid \Lambda_m \in S \} \) where \( (\Lambda_m)_{m=1}^n \) are the matrix eigenvalues. Then

\[
\frac{N_n(S)}{n} \xrightarrow{a.s.} \int_S \frac{1}{2\pi \sigma^2} \sqrt{4\sigma^2 - \lambda^2} \, d\lambda
\]

for \( S \in \mathcal{B}(\mathbb{R}) \) as \( n \to +\infty \).

This theorem generalizes the previous weak version. It actually claims that the portion of eigenvalues in some Borel measurable set converges to semicircle distribution applied on that set. Therefore, as well as the distribution mixture of matrix eigenvalues, its empirical version converges to the semicircle distribution too. Such a result is actually well known for iid random variables. Its proof is simply accomplished by the application of strong law of large numbers. Nevertheless, the eigenvalues of random hermitian matrices are usually heavily dependent which makes the strong version of the Wigner semicircle law a very general concept.

Regarding the procedure of establishing the strong semicircle law, the problem is again transformed in to the convergence of moments. Apparently, it is an almost sure convergence this time. Particularly,

\[
\frac{1}{n} \int_{\mathbb{R}} \lambda^k \, dN_n(\lambda) \xrightarrow{a.s.} E(\Lambda)^k
\]

where \( \Lambda \) is the random variable of the semicircle distribution. Using the matrix properties and the results about the moments of distribution mixture of the matrix eigenvalues, the
convergence above can be rewritten as

\[
\frac{\left| \text{Tr}(W_n)^k - E(\text{Tr}(W_n)^k) \right|}{n^{k/2+1}} \xrightarrow{a.s.} 0.
\]

Almost sure convergence of random variables are mostly being solved using the Borel-Cantelli lemma. In this case, the lemma is useful too. Specifically, the convergence above holds if the statement

\[
\sum_{n=1}^{+\infty} P \left( \left| \text{Tr}(W_n)^k - E(\text{Tr}(W_n)^k) \right| > \varepsilon n^{k/2+1} \right) < +\infty
\]

is true. The summands in this form are somewhat cumbersome to work with. Applying Chebyshev’s inequality, they can be bounded by the following way

\[
P \left( \left| \text{Tr}(W_n)^k - E(\text{Tr}(W_n)^k) \right| > \varepsilon n^{k/2+1} \right) \leq \frac{\text{Var}(\text{Tr}(W_n)^k)}{\varepsilon^2 n^{k+2}}. \quad (4)
\]

The variance of the matrix powers can be handled in a similar manner as in the proof of the weak semicircle law. After writing it out as a huge sum, the representation of the summands as some cycles in a graph allows to determine its behavior as

\[
\text{Var}(\text{Tr}(W_n)^k) = O(n^k) \quad n \to +\infty. \quad (5)
\]

Plugging this back to the expression (5) results in the sum in (4) to be convergent. According to the Borel-Cantelli lemma and Carleman continuity theorem, the proof of the strong version of the Wigner semicircle law is thereby concluded.

**Conclusion**

The main purpose of the research project was studying the Wigner semicircle law, one of the most famous and important theorems in the random matrix theory. The research resulted in providing a comprehensive, rigorous, but at the same time clear source summarizing the results about the theorem. The work was written to be self-contained and accordingly, there was an effort to include all the related calculations and derivations along with graphical representations where appropriate.

The author would like to continue in studying the Wigner semicircle law and related concepts. Specifically, there is an interest in understanding another famous proof of the theorem using Stieltjes transform. Apart from that, author would also like to immerse in the theory of free probability which describes the Wigner semicircle law from a very general point of view.

**Literature**

