EXACT MARKOVIAN EQUATIONS FOR COMPETITIVE EPIDEMIC SPREADING ON MULTI-LAYERED NETWORKS

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ABSTRACT. We derive exact Markov equations for $SIS$ epidemic spreading on a single network and $SIS_1SIS_2S$ competitive epidemic spreading on a multi-layered network. Moreover, we interpret these equations in terms of the graph Laplacian.

1. SIS Model

In this section we fill in the details of an argument to be found in [VanMi]. Given a graph $G = (V, E)$, consider a state vector $w = (w_i)_{i \in V}$ where $w_i = 0$ or 1 depending on whether the node $i$ is Susceptible or Infected. This vector will depend on time $t$, so we are interested in the process $w(t)$. Susceptible nodes become infected at a rate that is proportional to the number of infected neighbors, and infected nodes start a curing process that follows a Poisson distribution with a given rate. In formulas,

(1.1) \[ P[w_i(t + \Delta t) = 1 \mid w_i(t) = 0, w(t)] = \beta \sum_{j \in \mathcal{N}_i} a_{ij} w_j(t) \Delta t + o(\Delta t) \]

(1.2) \[ P[w_i(t + \Delta t) = 0 \mid w_i(t) = 1, w(t)] = \delta \Delta t + o(\Delta t) \]

These two transitions determine two more, namely

(1.3) \[ P[w_i(t + \Delta t) = 0 \mid w_i(t) = 0, w(t)] = 1 - \beta \sum_{j \in \mathcal{N}_i} a_{ij} w_j(t) \Delta t + o(\Delta t) \]

(1.4) \[ P[w_i(t + \Delta t) = 1 \mid w_i(t) = 1, w(t)] = 1 - \delta \Delta t + o(\Delta t) \]

Therefore, by the law of total probability,

\[ \mathbb{E}[w_i(t + \Delta t) \mid w(t)] = \mathbb{E}[w_i(t + \Delta t) \mid w(t), w_i(t) = 0] \mathbb{E}[1 - w_i(t) \mid w(t)] + \mathbb{E}[w_i(t + \Delta t) \mid w(t), w_i(t) = 1] \mathbb{E}[w_i(t) \mid w(t)] = (1 - w_i(t)) \beta \sum_{j \in \mathcal{N}_i} a_{ij} w_j(t) \Delta t + w_i(t)(1 - \delta \Delta t) + o(\Delta t) \]

where the last step follows from (1.1) and (1.4), and by the property of "perfect information" for conditional expectation.
Taking expectations, using the law of total expectation, unfolding and gathering terms, we get
\[
\mathbb{E} \left[ w_i(t + \Delta t) - w_i(t) \right] = \mathbb{E} \left[ -\delta w_i(t) + (1 - w_i(t)) \beta \sum a_{ij} w_j(t) \right]
\]
Letting $\Delta t \to 0$, the SIS governing equation is
\[
(1.5) \quad \frac{d\mathbb{E}[w_j(t)]}{dt} = \mathbb{E} \left[ -\delta w_j + (1 - w_j) \beta \sum_{k=1}^{N} a_{kj} w_k \right]
\]
The last term in (1.5) becomes
\[
(1 - w_j) \beta \sum_{k=1}^{N} a_{kj} w_k = \beta \sum_{k=1}^{N} a_{kj} w_k - \beta \sum_{k=1}^{N} a_{kj} w_k w_j
\]
Summing over all the nodes $j$:
\[
(1.6) \quad \frac{d\mathbb{E}[\sum_{j=1}^{N} w_j]}{dt} = \mathbb{E} \left[ -\delta \sum_{j=1}^{N} w_j + \beta \sum_{j=1}^{N} \sum_{k=1}^{N} a_{kj} w_k - \beta \sum_{j=1}^{N} \sum_{k=1}^{N} a_{kj} w_k w_j \right]
\]
Writing $Z = N^{-1} \sum_{j=1}^{N} w_j$ for the average number of infections, and $y = \mathbb{E}[Z]$ the expected average number of infections, (1.6) becomes
\[
(1.7) \quad \frac{dy}{dt} = -\delta y + \frac{\beta}{N} \mathbb{E} \left[ w^T Au - w^T w \right],
\]
where $u = [1 \cdots 1]^T$ is the “constant” vector. Note that $Au = d$ where $d(j) = \text{deg}(j)$ is the degree of node $j$ in graph $A$. Write $D = \text{diag}(d)$ for the diagonal matrix with the node degrees. Then, since $w_j$ is either 0 or 1,
\[
w^T Au = w^T d = w^T Dw = w^T w.
\]
So letting $L = D - A$ be the combinatorial Laplacian, we get that
\[
w^T Au - w^T Aw = w^T Dw - w^T Aw = w^T Lw.
\]
Then (1.7) says that
\[
(1.8) \quad \frac{dy}{dt} = -\delta y + \frac{\beta}{N} \mathbb{E} \left[ w^T Lw \right].
\]

2. Perimeter Evolution

Assume that an orientation has been chosen for every edge. Define the $m \times n$ incidence matrix
\[
B(e, x) = \begin{cases} 
1 & \text{if } \exists y \sim x \text{ such that } e = (y, x), \\
-1 & \text{if } \exists y \sim x \text{ such that } e = (x, y), \\
0 & \text{else}
\end{cases}
\]
We think of $B$ as a “gradient” operator that turns functions defined on $V$ into functions defined on $E$ by assigning the end-points difference to each edge. Then the transpose $B^T$
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is a “divergence” operator that turns functions defined on the edges to ones defined on the vertices.

Moreover, the Laplacian can be written as in the continuous case as the “divergence of the gradient”:

\[ L = B^T B. \]

To see this, write \( L(x,y) = \sum_{e \in E} B^T(x,e)B(e,y) = \sum_e B(e,x)B(e,y), \) and note that if \( x = y, \) then \( B(e,x)B(e,y) = 1 \) for every edge incident at \( x \) and is zero otherwise, so we get \( \text{deg}(x); \) but if \( x \neq y \) and \( x \sim y, \) then \( B(e,x)B(e,y) = -1 \) for the edge between \( x \) and \( y \) and zero otherwise, so we get \(-1.\)

The matrix \( B \) is sometime referred to as the “square root” of the Laplacian \( L. \) Given a function \( h \) on \( V, \) let \( \nabla h(e) = |h(y) - h(x)| \) if \( e \) connects \( x \) to \( y. \) Then the “gradient norm” of \( h \) is given by

\[ h^T L h = h^T B^T B h = (Bh)^T (Bh) = B h = \sum_{e \in E} |(\nabla h)(e)|^2. \]

In our case, the vertex set \( V \) is split into two subsets \( S \) and \( I \) and \( w \) is the indicator function of the set \( I. \) This partition defines a subset of edges called the “interface” (or edge-boundary), \( \partial I, \) consisting of all the edges that connect a vertex in \( S \) with a vertex in \( I. \) With this in mind, the quadratic form

\[ w^T L w = \sum_{e \in E} |(\nabla w)(e)|^2 = \sum_{e \in \partial I} 1 = |\partial I| \]

counts the number of edges in the interface \( \partial I. \)

So equation (1.8) becomes

\[ \frac{dy}{dt} = -\delta y + \beta N \mathbb{E}[\partial I]. \]

This says that the size of the perimeter of the infection determines how fast the average infections are growing, hence the term perimeter evolution.

3. COMPETITIVE SI1SI2S MODEL

In the competitive \( SI_1SI_2S \) model each node in \( V \) is either Susceptible (\( S \)), infected by virus 1 (\( I_1 \)) or infected by virus 2 (\( I_2 \)). For \( i = 1, 2, \) let \( w^{(i)} = (1_{\{w_j = i\}}) \) be the indicator function of the set of nodes infected by virus \( i. \)

\[
\begin{align*}
\frac{d\mathbb{E}[w^{(1)}]}{dt} &= \mathbb{E} \left[ -\delta_1 w^{(1)} + (1 - w^{(1)} - w^{(2)}) \beta_1 \sum_{k=1}^N a_{kj} w^{(1)} \right] \\
\frac{d\mathbb{E}[w^{(2)}]}{dt} &= \mathbb{E} \left[ -\delta_2 w^{(2)} + (1 - w^{(1)} - w^{(2)}) \beta_2 \sum_{k=1}^N b_{kj} w^{(2)} \right]
\end{align*}
\]

(3.1)

Unfolding, the last term in the first equation of (3.1) becomes
\[ (1 - w_j^{(1)} - w_j^{(2)}) \beta_1 \sum_{k=1}^{N} a_{kj} w_k^{(1)} = \beta_1 \sum_{k=1}^{N} a_{kj} w_k^{(1)} - \beta_1 \sum_{j=1}^{N} \sum_{k=1}^{N} a_{kj} w_k^{(1)} w_j^{(1)} - \beta_1 \sum_{k=1}^{N} a_{kj} w_k^{(1)} w_j^{(2)} \]

And summing over all nodes:

\[
\frac{d\mathbb{E} \left( \sum_{j=1}^{N} w_j^{(1)} \right)}{dt} = \mathbb{E} \left[ -\delta_1 \sum_{j=1}^{N} w_j^{(1)} + \beta_1 \sum_{j=1}^{N} \sum_{k=1}^{N} a_{kj} w_k^{(1)} - \beta_1 \sum_{j=1}^{N} \sum_{k=1}^{N} a_{kj} w_k^{(1)} w_j^{(1)} \right] 
\]

\[
= \mathbb{E} \left[ -\delta_1 \sum_{j=1}^{N} w_j^{(1)} + \beta_1 \sum_{j=1}^{N} \sum_{k=1}^{N} a_{kj} w_k^{(1)} w_j^{(2)} \right] 
\]

Let \( z^{(i)} = N^{-1} \sum_{j=1}^{N} w_j^{(i)} \) be the fraction of nodes infected by virus \( i \); and let \( y^{(i)} = \mathbb{E}[Z^{(i)}] \). Then (3.2) becomes in matrix notation:

\[
\frac{dy^{(i)}}{dt} = -\delta_1 y^{(i)} + \frac{\beta_1}{N} \mathbb{E} \left[ (w^{(1)})^T Au - (w^{(1)})^T Aw^{(1)} - (w^{(1)})^T Aw^{(2)} \right],
\]

where \( u = [1 \cdots 1]^T \) is the "constant" vector. Note that \( Au = d \) where \( d_j = \text{deg}(j) \) is the degree of node \( j \) in graph \( A \). Write \( D = \text{diag}(d_j) \) for the diagonal matrix with the node degrees. Then, since \( w_j^{(1)} \) is either 0 or 1,

\[(w^{(1)})^T Au = (w^{(1)})^T d = (w^{(1)})^T Du = (w^{(1)})^T Dw^{(1)}.
\]

So letting \( L = D - A \) be the combinatorial Laplacian we get that

\[(w^{(1)})^T Au - (w^{(1)})^T Aw^{(1)} = (w^{(1)})^T Aw^{(1)} - (w^{(1)})^T Aw^{(1)} = (w^{(1)})^T Lw^{(1)}.
\]

Then (3.3) is

\[
\frac{dy^{(1)}}{dt} = -\delta_1 y^{(1)} + \frac{\beta_1}{N} \mathbb{E} \left[ (w^{(1)})^T Lw^{(1)} - (w^{(1)})^T Aw^{(2)} \right],
\]

Likewise

\[
\frac{dy^{(2)}}{dt} = -\delta_2 y^{(2)} + \frac{\beta_2}{N} \mathbb{E} \left[ (w^{(2)})^T Lw^{(2)} - (w^{(2)})^T Bw^{(1)} \right].
\]

REFERENCES


[VanMi] Van Mieghem P., Exact Markovian SIR and SIS epidemics on networks and an upper bound for the epidemic threshold, Delft University of Technology, report 20140210, 2014.

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