THE $p$-HARMONIC TRANSFORM
BEYOND ITS NATURAL DOMAIN OF DEFINITION,
INTERPOLATION AND CONTINUITY.

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JOINT WORK WITH TADEUSZ IWANIEC

Abstract. To every vector field $f \in L^q(\Omega, \mathbb{R}^n)$, $\Omega \subset \mathbb{R}^n$, there corresponds a unique solution $u \in W_0^{1,p}(\Omega)$ of the $p$-harmonic equation
\[ \text{div} |\nabla u|^{p-2} \nabla u = \text{div} f, \]
where the exponents satisfy Hölder’s relation: $p, q > 1$ and $p + q = p \cdot q$. The $p$-harmonic transform assigns to $f$ the gradient of the solution.
\[ \mathcal{H}_p : L^q(\Omega, \mathbb{R}^n) \rightarrow L^p(\Omega, \mathbb{R}^n), \quad \mathcal{H}_p(f) = \nabla u \]

More general PDEs and the corresponding nonlinear transforms are also considered.
We are concerned with the continuous extension of the $p$-harmonic transform beyond this so-called natural domain of definition. Namely,
\[ \mathcal{H}_\lambda : L^{\lambda q}(\Omega, \mathbb{R}^n) \rightarrow L^{\lambda p}(\Omega, \mathbb{R}^n), \quad \text{for some parameters } \lambda \geq \max\left\{ \frac{1}{p}, \frac{1}{q} \right\} \]

First, we establish an Interpolation Theorem. Because of nonlinearity, this result requires substantial innovations of the familiar Marcinkiewicz ideas from the linear theory. Then we explicitly identify the so-called critical parameter $\lambda$ for which the existence, uniqueness and continuity of $\mathcal{H}_\lambda$ take place. Surprisingly, the uniqueness property in unbounded domains is lost when $\lambda$ exceeds the critical parameter. It is a little more surprising that the $n$-harmonic transform in unbounded domains, say $\Omega = \mathbb{R}^n$, cannot be extended to any Lebesgue space $L^s(\mathbb{R}^n, \mathbb{R}^n)$, with $s > n$. In other words, the critical parameter is equal to 1 in this case.