Exercises Set # 5

Solve on space \( \mathbb{R}^n \), \( n \geq 1 \)

1. Show that a function \( u \) is weakly differentiable in a domain \( \Omega \) if and only if it is weakly differentiable in a neighborhood of every point in \( \Omega \).

2. Let \( \alpha, \beta \) be multi-indices and \( u \) be a locally integrable function on a domain. Show that provided any two of the weak derivatives \( u^{\alpha+\beta}, \partial^\alpha (\partial^\beta u), \partial^\beta (\partial^\alpha u) \) exist, they all exist and coincide a.e. in \( \Omega \).

3. Let \( \Omega \) be a domain containing the origin. Show that \( \delta(x) = |x|^{-\alpha} \in W^k(\mathbb{R}) \) provided \( k \in \mathbb{N} \).
\[ W^{k}(\Omega) = \{ u \in L^\infty(\Omega) \mid \text{u is \(k\)-times weakly differentiable} \} \]

Functions $C^{0,1}(\Omega) \subset W^{1,1}(\Omega)$, $u$ is weakly differentiable with locally bounded weak derivatives.

$\left( C^{0,1}(\Omega) = \{ \phi \in L^\infty(\Omega) \mid \phi \text{ is continuous,} \right.$

and $|\phi(x) - \phi(y)| \leq c|x - y|$, for $x, y \in \Omega \}$.

5. Show that if $u \in W^{1,1}(\Omega)$ and $\omega \subset \subset \Omega$, $\omega, \Omega$ open sets, then

\[ \frac{u(x + te_1) - u(x)}{t} \to \partial_1 u \in L^1(\omega), \quad 1 \geq t \geq 0. \]

6. Show that if $v_n \to \omega$ in $L^1(\Omega)$ and $\Omega$ bounded.
then \( u \in W^{1,0}(\Omega) \).

7. Let \( \Omega \) be a domain in \( \mathbb{R}^n \). The total variation of a function \( u \in L^1(\Omega) \) is defined by

\[
\sum_{1 \leq i \leq n} |D_i u| := \sup \left\{ \int_\Omega \sum_{1 \leq i \leq n} \left| \frac{\partial u}{\partial x_i} \right| \, dx : \alpha \in C_0^1(\Omega) \right\}
\]

\( |\alpha| \leq 1 \)

\( \omega : \Omega \rightarrow \mathbb{R}^n \), \( \omega = (\omega_1, \ldots, \omega_n) \), \( \alpha \in C_0^1(\Omega) \), and \( D_i \alpha := \sum_{j=1}^n \frac{\partial \alpha_j}{\partial x_i} \).

Show that the space \( BV(\Omega) \) of functions of finite total variation is a Banach space under the norm

\[
||u||_{BV(\Omega)} = ||u||_{L^1(\Omega)} + \sum_{1 \leq i \leq n} |D_i u|
\]

\( \diamondsuit \) Note: this does not mean that \( ||u|| \) is a function!! The left hand side in \( \diamondsuit \) is just
a) Let $B$ be the unit ball in $W^{1,p}(\mathbb{R}^n)$, i.e.,

$b$) Show that $B|_w$, $w \subset \mathbb{R}^n$, is relatively compact in $L^p(w)$. 

Show that if $(u_m) \subset W^{1,p}(\mathbb{R}^n)$ is bounded in $W^{1,p}(\mathbb{R}^n)$, and $1 \leq p < \infty$, then for an arbitrary open subset $\Omega_0$ of $\mathbb{R}^n$, there is a subsequence $(u_{m_k})$ of $(u_m)$ s.t. $u_{m_k}(x)$ converges, for a.e. $x \in \Omega_0$. 