Exercises List #3

Sobolev Spaces

1. \(1 \leq p \leq \infty\) show that a piecewise continuous function on \([a,b]\) belongs to \(W^{1,p}(a,b))\)

2. Let \(1 < p < \infty\), and \((u_n)_{n \in \mathbb{N}} \in W^{1,p}(a,b))\), \(u \in L^p(a,b))\), such that \(u_n \rightharpoonup u\) in \(L^p(a,b))\) and \((u_n)_{n \in \mathbb{N}}\) is bounded in \(L^p(a,b))\). Show that \(u \in W^{1,p}(a,b))\) and that \(u_n \rightharpoonup u\) of \(u_n \rightharpoonup u\) in \(W^{1,p}(a,b))\).

3. Assume \(I = (a,b)\) is bounded. Show that the polynomials are dense in \(W^{1,p}(I)\) for
1. \( p < \infty \). What happens if \( p = \infty \)?

4. Show that every Lipschitz function on \((a, b)\) is a.e. differentiable on \((a, b)\).

5. Show that if \( C^\infty \) is dense in \( W^{1,1}(I) \) then \( I = \mathbb{R} \).

6. Consider the Neumann problem

\[
\begin{cases}
-uu'' + uu' = f & \text{in } (a, b) \\
u(a) = u(b) = 0
\end{cases}
\]

with \( f \in L^2([a, b]) \).

a) Show that if \( u \in C^2((a, b)) \cap C^1([a, b]) \) is a solution of the Neumann problem then \( u \) satisfies the weak formulation

\[
\int_a^b u'v' \, dx + \int_a^b uu' \, dx = \int_a^b fv \, dx
\]
for any $\varphi \in C^1((a,b)) \cap C([a,b])$.

6) Show that if $f \in L^2((a,b))$, then $u \in H^1((a,b))$ is the weak solution of the Dirichlet problem.

7. Prove the Poincaré inequality: \exists C > 0

8.1. $\|u\|_{L^2((a,b))} \leq C \| \nabla u \|_{L^2((a,b))}$ for any $u \in W^{1,2}((a,b))$ s.t. $\int_a^b u = 0$.

8. Show that the function $f(x) = |x|^{-\alpha}$ lies $k$ weak derivatives in $\mathbb{R}^n$ provided $k + \alpha < n$. 

\[ \exists C > 0 \]