Weak topologies I

(a) $E$ a Banach space, $\phi: E \to \mathbb{R} (\pi E)$ linear. Then

\[ \phi \text{ is continuous } \iff \phi: (E, \sigma (E, E^*)) \to \mathbb{R} \text{ continuous.} \]

(b) $E, F$ Banach, $T: E \to F$ linear.

$T$ is continuous $\iff T: (E, \sigma (E, E^*)) \to (F, F^*)$ is continuous.

2. $E$ a Banach space, $(x_n)_{n \in \mathbb{N}} \subseteq E$, $x \in E$.

If $x_n \to x$ then there is a sequence of convex combinations of the $x_n$ that converges strongly to $x$. (This is Mazur's Theorem.)
3. \( E \) be normed, \((x_n)_{n \in \mathbb{N}} \) in \( E \)

\( x_n \) converges in \( E \) \( \implies \) \( x_n \) converges weakly, and uniformly in \( \varepsilon \) \( \forall \varepsilon \in E' : \| \varepsilon \| = 1 \)

4. \( \varepsilon \) set \( 1 < p < \infty \) and, in \( L^p \), \( e^n \) given by

\[ (e^n)_k = e^n_k \] - show that

\( e^n \) \( \rightharpoonup \) \( 0 \) \( \implies \) \( e^n \to 0 \), \( e^n \geq 0 \)

5. \( 1 < p < \infty \), \( \varepsilon^n \), \( \varepsilon \in E^p \). Then

\[ \varepsilon^n \to \varepsilon \implies \sup\{\| x^n \|_{L^p} \} \to \sup\{\| x \|_{L^p} \} \]

6. \( 1 < p < \infty \), \( (\Omega_n, \chi) \in L^p(\mathbb{R}, \mathbb{R}) \)

\( \chi \) then \( (\Omega_n \to \chi) \) \( \sup\{\| x \|_{L^p} \} \to 1 \)

\( \int_{0}^{a} (\varepsilon_n(t)) \, dt \to \int_{0}^{a} \chi(t) \, dt \quad \forall \varepsilon \in \mathbb{R}, t \)
b) Show that \( q_n(t) = \sin(n \pi t) \) is \( L^2([0,1]) \) converges weakly to 0, but does not converge strongly to 0.