Exercise Set #11

Hilbert Spaces

1. Show that the following are Hilbert spaces:

   a) $\mathbb{C}^n$, with $(x,y) = \sum_{k=1}^{n} x_k \bar{y}_k$

   b) $\mathbb{R}^n$, with $(x,y) = \sum_{k=1}^{n} x_k y_k$

   c) $l^2$, with $(x,y) = \sum_{k=1}^{\infty} x_k \bar{y}_k$ (convergent series)

   d) $l^2(\mathbb{R})$, $x \in l^2(\mathbb{R})$ measurable, with

   $(f,g) = \int f(x) g(x) dx$

2. If $H$ is a Hilbert space over $\mathbb{C}$, the polarization formula:

   
   $$(xy) = \frac{1}{4} \left( \|x+y\|^2 - \|x-y\|^2 + i(\|x+iy\|^2 - \|x-iy\|^2) \right)$$

   holds.

3. a) $(E, \| \cdot \|)$ a Banach space. Show that

   $\| \cdot \|$ is induced by an inner product iff

   $\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$ for all $x, y \in E$. 

   b) $l^2$ verifies the parallelogram identity
b) \((L^p, \| \cdot \|_p)\), \(p \geq 2\), and \((C_{00}, \| \cdot \|_\infty)\)
are met Hilbert spaces.

4. A Hilbert space, \(E_n\), spanned by \(n\) orthonormal vectors.

The following are equivalent:

a) \(E_n\) is maximal orthonormal.

b) if \(\exists x \in H\), \(x \perp E_n\), then \(x = 0\).

5. Gram-Schmidt orthogonalization process:

\(H\) a Hilbert space, \(E_n\) is a l.i. subset of \(H\) such that \(S = \langle E_n \rangle\) is dense in \(H\).

a) let \(e_1 = \frac{b_1}{\|b_1\|} \) be defined. \(e_n\), set

\[
e_{n+1} = \frac{b_{n+1} - \sum_{k=1}^{n} (b_{n+1}, e_k)e_k}{\| b_{n+1} - \sum_{k=1}^{n} (b_{n+1}, e_k)e_k \|}
\]

show that \(E_n\) is a base of \(H\).

b) let \(\{f_n\}\) be an orthonormal set s.t.

\(H_n = \langle f_1, \ldots, f_n \rangle, \| f_n \| \to 0, \) s.t.
\[ p_n = \frac{1}{2} \sum_{i=1}^{n} \lambda_i b_i \quad \text{then} \quad p_n \to x_0 \\ \text{en, with} \]

\[ x_n \in \mathbb{C}, \quad \|x_n\| = 1 \quad \text{for} \ n. \]

6. \( H \) a Hilbert space, \( x_n, x \in \mathbb{H} \) - then

a. \( x_n \xrightarrow{\omega x} \iff (x_n, y) \to (x, y) \quad \forall y \in \mathbb{H} \)

b. If \( \{x_n\}_{n=1}^{\infty} \) is a sequence of \( H \) then \( x_n \xrightarrow{\omega x} x \)

c. \( x_n \xrightarrow{\omega x} \) and \( \|x_n\| \to 0 \) then \( x_n \to x \)

d. Let \( \{x_n\}_{n=1}^{\infty} \) be a sequence of \( H \) - then \( x_n \xrightarrow{\omega x} x \iff (x_n, y) \to (x, y) \quad \forall y \in \mathbb{H} \)

e. \( x_n \xrightarrow{\omega x} x \iff \exists (x_{n_k})_{k=1}^{\infty} \quad \text{s.t.} \quad \text{the seq. of arithmetic means} \)

\[ z_k = \frac{1}{k} \left( x_{n_1} + \cdots + x_{n_k} \right) \to x \]

7. Let \( \{x_n\}_{n=1}^{\infty} \) be an orthogonal set in the Hilbert \( H \) - the following are equivalent:
e) \( \sum_{n=1}^{\infty} x_n \) converges

b) \( \sum_{n=1}^{\infty} x_n \) converges weakly

c) \( \sum_{n=1}^{\infty} \|x_n\|_2 \) converges

8. a) A Hilbert space \( H \) is separable if every orthonormal system in \( H \) is at most countable.

b) If \( \#(I) > \aleph_0 \), \( l^2(I) \) is not separable.

9. a) If a Hilbert space, \( DC H \), \( \subset \mathbb{C}^N \), the subspace generated by \( D \) is dense in \( H \).

(\( \langle x,y \rangle = 0 \ \forall y \in D = x = 0 \)).

b) Let \( S = \{ x \in l^2 : \sum_{n=1}^{\infty} \varphi_n = 0 \} \subset l^2 \).

Show that \( S \) is dense in \( l^2 \).

10. Let \( S, T \subset H \) be closed subspaces; \( S \cap T \).

Show that \( S \cap T \) is closed.
11. Let $H, K$ Hilbert spaces. In $H \times K$ we define
\[(h_1, k_1), (h_2, k_2) = (h_1, h_2)_H + (k_1, k_2)_K - \text{Show that } (H \times K, (\cdot, \cdot)) \text{ is a Hilbert space,}
\end{equation}
and that $H \times 0$ and $0 \times K$ are closed and mutually orthogonal in $H \times K$.

12. Let a Hilbert space $H$, $S \subseteq H$ a closed subspace. Show that
\begin{enumerate}
    \item $\exists x \in H : x \perp S$
    \item $S^\perp = \{ x \in H : x \perp S \}$.
    \item $S$ is a closed subspace and $S \oplus S^\perp = H$.
    \item $(S^\perp)^\perp = S$
    \item Exhibit counterexamples showing that b) and c) do not hold if $S$ is not closed.
13. a) The set \( \{1, x, x^2, \ldots \} \) is l.i., and generates a dense subspace of the Hilbert space \( L^2([0,1]) \). Orthogonalizing by Gram-Schmidt we obtain \( \{u_n(x)\} \), which satisfies

\[
e_n(x) = \left( \frac{2n+1}{2} \right)^{1/2} \sqrt{n} \, p_n(x),
\]

with \( p_n(x) = \frac{1}{2^n n!} \left( \frac{d}{dx} \right)^n (x^2-1)^n \) the Legendre polynomials.

b) The set \( \{\sqrt{n} e^{-x^2} \, e_n(x) \, e^{-x^2} \} \), \( n \geq 0 \), is l.i. and generates a dense subspace in the (real) Hilbert space \( L^2((-\infty, \infty)) \). Orthogonalizing by Gram-Schmidt we obtain \( \{u_n(x)\} \), which satisfies

\[
e_n(x) = \frac{1}{\sqrt{\pi} \, 2^n n!} \, h_n(x) e^{-x^2}, \quad \text{with}
\]

\[
h_n(x) = \frac{1}{2^n n!} \left( \frac{d}{dx} \right)^n (x^2-1)^n.
\]
\[ F(x) = (-1)^n \frac{d^n}{dx^n} e^{-x^2} \text{ for Hermite polynomials.} \]

\[ H_n = 2n H_{n-1} \]

\[ (e^{2x})^n = e^{2nx} \]