First Synthesis Paper

One of the greatest benefits of my studies thus far in mathematics education is that it has reawakened a sense of introspection concerning my teaching practices. I am thinking more about what I do in the classroom than I have for many years. The following paper has been constructed from notes I have made as I read the articles assigned for this class and from practices and ideas that I have found useful during my experience teaching.

The following main topics are addressed in this paper:

What does it mean to understand mathematics?

How does mathematical understanding develop?

An example of how what I have outlined actually works.

How my theory can be used in instruction, assessment or research.

What does it mean to understand mathematics?

The word “understanding” is a difficult word to pin down. There are several considerations when defining an “understanding” of mathematics.

Mathematical understanding is certainly not the same for all levels of difficulty. The level of understanding will decrease as the level of complexity increases for any individual. There is a ceiling for every individual beyond which they cannot grasp concepts or understand the material further and this ceiling varies in height for each person. If this were not so mankind would have no unresolved problems left.

A person may exhibit understanding in one or more areas of mathematics but display little in other areas. This may be an effect of personal interest in some particular area of mathematics or the result of natural ability for certain types of mathematics. I have often noticed a decided affinity amongst high school students for either geometry or algebra but rarely for both. An extreme case would be the example of an idiot/normal savant. These individuals sometimes exhibit great dexterity in a single area of mathematics. This “understanding” is almost godlike within its scope.
Mathematics may also be understood in ways that cannot be assessed or verified. This type of understanding may come about when an individual realizes that mathematics fits reality much better than there is any reason to expect. It may manifest itself when observing the beauty of patterns, curves, or chaotic systems that have been mathematically described. Graphical representations of fractals or the precision with which physics describes nature also produce this type of mathematical understanding.

I find Skemp’s definition of mathematical understanding in the article *The Psychology of Learning Mathematics* the most compelling of all the definitions given by the various authors we have read so far in class, “To understand something means to assimilate it into an appropriate schema.”

I have spent many hours looking at the same pieces of information, from either textbooks or my own calculations, and suddenly some small change in the way I look at the data makes it all clear. To illustrate I will relate my experiences in Chemistry. Through no fault of my instructors I did not grasp the concept of a mole. After thinking about the matter for some time and doing additional reading I came across an author who calculated that a mole of doughnuts would cover the earth’s surface to a depth of about 200 miles. It was this revelation that a mole was simply a number of things, exactly like the idea of a dozen that transformed my understanding of the concept. From that point on I was able to deal with the mathematical portion of chemistry competently and forge a very profitable relationship with another student who did not understand the mathematics but was a whiz at actually doing experiments.

Below I have attempted a compilation of different levels of mathematical understanding from shallow to deep understanding. In each case a reconstruction of schemas is necessary to understand at the next level. The arrangement of levels is certainly fluid and much discussion is possible, yet I feel that the trend is correct.

1) Understanding mathematics means being able to perform routine calculations as they are needed on a day-to-day basis.

2) Understanding mathematics means being able to quickly and efficiently do complex mathematical tasks that were explicitly taught or are demanded by job requirements.

3) Understanding mathematics means being able to perceive patterns and solve “mathematical puzzles”. Mensa tests fall in here somewhere.

4) Understanding mathematics means being competent to perform mathematically over a wide range of mathematics.
5) Understanding mathematics means being able to construct computer programs to perform various mathematical tasks.

6) Understanding mathematics means being able to apply previous mathematical knowledge and techniques to new situations.

7) Understanding mathematics means being able to see relationships between different areas of mathematics. The ‘new math’ essentially demanded this level of understanding from a population of students largely incapable of this depth of understanding at any age.

8) Understanding mathematics means being able to appreciate the germination, growth and final maturity of mathematical concepts historically. To appreciate the overarching framework of a concept throughout history is not the trivial matter that many students assume it to be.

9) Understanding mathematics means being able to teach the subject to another person from multiple viewpoints.

10) Understanding mathematics means being able to invent new mathematics. Western society has placed this at the pinnacle of understanding by demanding that education’s highest degrees be given only to individuals who demonstrate that they have created new knowledge. This is perhaps the equivalent of a stockbroker actually becoming wealthy in the stock market.

Society will determine the level at which individuals need to understand mathematics. If the level is set too low society will suffer economically and culturally. Set the level at an unrealistically high level and the system will turn and devour itself in frustration and recrimination.

My experience tells me that the United States will not tolerate the actual level set past level number three. The current Kansas State Standards aim for a goal around number seven but test at level three. The state assessment has many characteristics in common with puzzle work and probably is a better indicator of aptitude than mathematical understanding.

Many of the countries against whom we compare ourselves have societies that set the level of understanding at about the fourth level. Exit exams are common in most of the countries that do well on international exams and these tests demand a broad competency in mathematics. The booklet “What Students Abroad Are Expected to Know About Mathematics” published by the American Federation of Teachers is enlightening in this regard.

Teaching for understanding at levels above four is likely unrealistic for the general population. Only students whose intelligence and/or work ethic are superior are capable of attaining higher levels of understanding. Any attempt, as
is now the fashion, of expecting all children to reach deep levels mathematical understanding will only yield self-deception on the part of the educational community and ultimately societal disappointment and a move back to expectations of only the first level of understanding.

**How does mathematical understanding develop?**

In my concept of understanding mathematics the above question includes, What causes schema reconstruction and assimilation to take place? Skemp describes an event called a “crisis of learning” that causes schema reconstruction. Below are four areas that can generate the necessary “crisis of learning”.

1. **Motivation**

   Students must have strong motivation to begin the learning process. Without this as a basis for education dismal results are achieved. Usually this is provided by society/schools in the form of negative and positive consequences. Since negative consequences are on the whole much cheaper and more efficient they are the most common. Most societies that the United States compares itself to regarding mathematical understanding of students provide far greater negative consequences for failure to learn. Considering that a person’s actions are essentially driven by negative consequences from birth to death it seems absurd to deemphasize this powerful tool in education. The $\pi$ in the sky ideal where students will be motivated to learn if only the instruction was presented correctly as advocated by Dubinsky bears no relationship to reality that I can see.

   There is an inverse relationship between the need for good teachers and the level of student motivation. On this basis, there will be no shortage of teaching opportunities in the United States. The United States has limited the effectiveness of its educational institutions by not insisting that teachers be intellectually capable and adequately prepared. Lee Shulman addressed the lack of emphasis on content preparation but he does not go far enough. Teachers that are not intellectually capable themselves can be given extensive content and pedagogical training and still not be effective.

2. **Teaching and Time**

   Teachers are essential to the learning process in most situations. They provoke a “crisis of learning” by directing student activities in a manner that will emphasize the inadequacies of old schemas. Only a teacher who has actually gone through a number of schema reconstructions relating to mathematics will be able to guide students correctly.
Time spent with a teacher is also crucial. It is necessary for the student to spend more than a semester with an instructor for any large impact to generally occur. Our current system does a very poor job of mentoring students through their educational careers.

3. Pure and Applied Viewpoints

Mathematics must be taught in close connection to the physical world as well as from a formal viewpoint if deeper levels of understanding are to be expected. With Morris Kline I believe that teaching mathematics without close attachment to the physical world is of little use for the vast majority of students. This connection drives types of mathematical thought much sought after by society including problem solving and modeling.

4. Practice

Perhaps the largest factor in reconstructing schemas is practice. Repeated practice with unfamiliar mathematics forces the “crisis of learning” to occur by keeping the irritation directly in front of the student. The student must now deal with the irritation until he is willing to do something about it. Far from being a detriment, practice is essential to understanding mathematics. I am not advocating the same amount of practice for every student. There will be a few students who will need less practice to construct an appropriate schema as there will be some who need far more practice than society can afford to provide. The mandatory assignment and grading of homework should happen daily at the lower levels of mathematics instruction, more infrequently at middle levels and not at all in upper level classes. If homework is graded at the upper levels we deny students the opportunity to learn the important lesson of self-discipline.

An example of how what I have outlined actually works.

My theory prescribes the following:

1. Motivation of Students
2. Quality Teachers and Time
3. Connect Mathematics to the Physical World
4. Circular Practice

I taught Algebra III (pre-Calculus), Calculus, Physics, and Mathematical Programming to high school students for more than a decade in the small southwestern Kansas town of Lakin. This is perhaps as close as I have ever come to my ideal teaching situation. I had the same students for 4 years of
mathematics and 1 year each of Physics and Mathematical Programming. The senior year I had my students for 3 hours a day.

The motivation of my students was no better than most students in our country. I did have the support of parents for demanding coursework and grading. This took time and a public relations campaign to build up but was well worth the effort.

As a teacher I was well prepared to teach all of the courses listed. My undergraduate major was physics and my master’s degree was in mathematics. I know well the benefits of better content knowledge as after completing my mathematics degree I realized that I had been short changing my students in my mathematics classes. I could now put their mathematical experiences into perspective for them and I knew what to emphasize, de-emphasize and discard. My teaching became much more efficient. I am also an intellectually capable teacher and so was able to relate to the students that I was teaching. I could understand them because I had been in their shoes myself.

I linked the physics and mathematics tightly together at every opportunity. Physics provided much of the motivation for learning mathematics and mathematics ensured that all the tools for success in physics were present. To complete the connection the students were also enrolled in Mathematical Programming. Both mathematics and physics provided problems for programmable calculator solution and students were understanding mathematics at level five. I found one of Dubinsky’s comments particularly relevant to my theory of instruction. “We have found activities with computers to be a major source of student experiences that are very helpful in fostering reflective abstraction”, Ed Dubinsky, Reflective Abstraction in Advanced Mathematical Thinking.

The curriculum that I chose for both mathematics and physics was Saxon’s Incremental Development. The constant circular (not spiral) review of the homework assignments did exactly the type of crisis creating that provoked students to change schemas. During the junior and senior years I strongly encouraged students to do their daily homework but did not grade the work. They did as little or as much as they desired. Many abused this freedom in the first few weeks but almost all learned to regulate their study habits so that they could succeed.

The results of this program of instruction were very good. Comparing ACT scores before this program and afterwards showed a dramatic jump upward in scores. The mathematics and science scores were consistently 10-20 percentage points higher than other sub scores. These students were also successful at postsecondary schools. This program probably turned out more students that pursued mathematical or physical science related degrees than any school of its size in the state.
How my theory can be used in instruction, assessment or research.

What must be done to motivate students.

The motivation of students is largely a function of our society's willingness to put the majority of the responsibility for learning on the student. At this point American society believes ever more completely that modern teaching techniques are the answer to our poor performance compared to other world powers. No matter that these techniques cannot be shown to work any better than normal or even poor instruction. To be fair to the public the teaching community has given the impression that large gains have been made in the science of learning. Witness the growth of special education programs where assorted “learning problems” are supposedly addressed. Students are constantly categorized by trivial tests, (right/left brain, learning styles etc) administered under the most shaky conditions. Students come to believe that learning is certainly not something they are responsible for but depends on some random condition assigned to them at birth.

The American educational establishment has further persuaded the public that any high stakes testing is an unnecessary evil. Students will not be motivated to prepare and perform at their best if there are no consequences for bad test results. As an example, the Kansas State Assessments actually have very little reliability because there are no individual consequences for poor performance. Ironically the blame is placed on the teachers and school districts that cannot substantially influence the value the students place on the assessment. A counterexample would be the Texas State exit exams or the Chicago school districts’ passing exams. Scores have improved dramatically in both instances where motivation was provided.

What must be done to have better teachers.

As regards teachers and preparation our society must stop drawing its raw material for teaching from the bottom of the academic barrel. Government and institutional studies have shown that teachers have the lowest ACT scores of any degree. If quality students were attracted to the teaching profession by good working conditions and school systems were not allowed to hire future teachers without proof of their intelligence the problem of content knowledge would be fixed as a direct consequence. It would also be very good for our educational system to divorce itself from athletics and turn that job over to the local municipality. This is the method used successfully in many countries. The distraction of time away from school, the cost of equipment and personnel, and the low academic quality of coaches attracted to the system would be alleviated.
What must be done to connect Mathematics with the content of other disciplines.

Much of the connection between mathematics and other disciplines needs to come from the other disciplines themselves. It is rare that other subjects demand mathematical treatment of their content. If schools were to mandate that all core subjects must include mathematics as part of the grading structure students would not be so quick to ask what mathematics is good for. I advocate a structure which places language and its use and mathematics at the top of the academic pile. All subjects would be required to demonstrate and grade the use of good language skills and mathematics competence.

This past spring I visited with an Olathe school district about an integrated aerospace/mathematics/physics program. Each subject was team taught with teachers from the other two disciplines. They were also doing this with other types of curriculum including graphical arts and computer programming. This is certainly a step in the right direction.

What must be done insure that correct and sufficient practice takes place.

There is ample evidence that shows that content delivered in small increments, circular practice, and frequent assessments actually produce learning gains. The Saxon Incremental Development approach is one curriculum that delivers this type of instruction. The mathematics educational establishment must stop frowning on well documented approaches to learning. The attitude that it must not be a good approach if it does not have room for many doctoral theses to be written it must be bad is not good for students.

I believe that all of the knowledge needed to have sound learning are and have been in place for some time. Until science is prepared to explain what actually happens to information as it is placed, retrieved and linked in the brain we would be better off seeing to it that students are motivated, taught by competent and intelligent teachers, understand the value and place of mathematics, and are given the correct type of practice. After we do understand how information is actually handled by the brain we will finally be able to just pour knowledge into students' brains as they do in the Matrix. That will be a sad yet wonderful day.