What it means to understand mathematics

How people learn has been an important question to me over the years that I have been teaching. I’ve read scores of books on teaching strategies and techniques as well as journals with unique and interesting approaches to the subject matter at hand. Despite all of this my questions still remained unanswered. I’ve come to the conclusion that it is because I have put the cart in front of the horse. To answer the question concerning how people learn, I must first consider what the final product looks like. It’s as if I’m trying to describe how to get somewhere but I don’t know where “somewhere” is. I might give directions that result in a successful journey, but most likely I would not. I would be missing the details that help the traveler know they are headed in the right direction and enables them to realize that they have actually “arrived”! This means that I need to start with what it means to understand mathematics.

Students understand mathematics when they have control over it at the level at which they are functioning. This means that the mathematics they are studying “makes sense” to them. They can describe not only how to perform the procedure, but how it works and why they would use it. When the mathematics students are studying is useful to them and they can recognize appropriate situations in which to apply it, then they most likely understand what they are doing. Students also show that they understand when they express their interpretations of the mathematics they are studying at a relational level. Rather than memorizing isolated bits of knowledge, recalling one skill independently of from the next, or repeatedly practicing one procedural disconnected from the other,
students are linking the many concepts and skills together. Students understand mathematics when they recall their prior knowledge and use it to build new concepts. In addition, they can transfer their knowledge to a new situation even if it is not like one they have experienced before. In general, they have the characteristics of an “expert” at that particular level.

I share the view of NCTM that control of mathematics involves conceptual understanding, procedural fluency, and factual knowledge. James Hiebert speaks of this as form and understanding. I see the conceptual part as what it means to understand mathematics and the form part as what it means to perform mathematics. You certainly can have one without the other. Analyzing my own learning has enabled me to experience these three components. I think the perfect argument for teachers studying higher-level mathematics, beyond what they will be teaching, is to study it as much for metacognitive purposes as for the purpose of learning more mathematics. I also see the argument for learning about the history of mathematics. You can see the struggle and the development of the concepts as well as the struggle to put it in a “form” that others can understand. Often the inability to understand what those before us have discovered leads us to the memorization of the form that they have left behind. For some they do this because they don’t know the difference. For people like me, I memorize procedures because I don’t have the time or the tools for making sense out of what I am learning. I do it because I’m scared I’m not smart enough to do mathematics at the level that involves “understanding” so I mimic those who are “smart enough”.

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How this understanding develops

In class we answered the question, “what sort of mathematical understanding is necessary for effective control of mathematical problem solving?” I think another interesting question that would help define what it means to understand mathematics would be, “what sort of control of mathematics is necessary to develop mathematical understanding?” What if we looked at it from the perspective of teaching students to control mathematics as a way of developing understanding? Consider the characteristics of people who understand mathematics. Those things tend to involve control in some fashion. Who is to say that the understanding has to come first? The following are issues that students can learn to control and by learning to control them understand mathematics.

Students need to recognize that acquiring knowledge is an individual process, in the sense that we organize our experiential work based on our individual interpretation of the experiences that we encounter. As a result, no two people have exactly the same knowledge and the knowledge that we do share is not defined or represented in exactly the same way. Students need to gain control over their knowledge and recognize their individual differences.

Linked with this is that understanding mathematics means that students are engaged in cognitive activity. Students need to have control over the active process of exploration to truly understand mathematics. Exploration and hands-on participation leads to the invention of ideas. Again, this exposes individual differences as people can perceive the same event differently. This is why it understanding mathematics is an active
process both in the process of experiencing the mathematics as well as sharing the interpretation with others to get a broader perspective. I share this view with Piaget who believed that students construct their knowledge through actions on the world. He believed that to understand is to invent. We generate our own rules and models and then adjust these to accommodate new information. Thinking can be taught. The metacognitive processes can be modeled, shared, discussed so that student can better control how their own mind works and as a result lead to a deeper understanding. Most importantly, thinking can not be done for a student.

Another “control” issue is allowing the mind to be stretched. The mind can be stretched beyond what it is independently capable of doing with assistance from another. Students can help or hinder this process. Vygotsky calls this the “Zone of Proximal Development”. There are three categories to Vygotsky’s zone. There are tasks that students can perform independently. There are those tasks that students can’t do even with help. And there are those tasks that lie in the middle. I call it the learner’s potential. This is why the educational system is so important. I actually teach my students about the zone. I explain to them that this is the purpose of assignments that I would first ask them to work on by themselves or with their peers. I want them to do everything they can do on their own. They will reach a point where they can’t find a way to go any further. This is where I step in to stretch them to learn more that they could do before. Students who won’t even try aren’t even aware of when they are in the zone. I want them in the zone because I know they are learning about things they didn’t know or think about before and this is where the best learning occurs. We can then push the bottom level up and those things that were beyond what students can do even with help become within their grasp.
eventually. This teaches students to use everything that they already know. The breakdown in the ability to reach potential is sometimes due to lack of motivation, poorly defined learning tasks, ill-prepared instructors, inappropriate lesson development, or insufficient assessment.

I heard a quote one time that compared studying mathematics to playing a basketball game. If you’re not sweating then you’re not in the game! For learners, this means that they will experience frustration and confusion and they have to learn to control how they sort this out and get answers to questions that they have. Learning also requires reflection on the experience, the environment, and on thoughts about them. Without reflection, the frustration and confusion is worthless. Learning takes time and patience, both for the learner and the teacher. So often students only see the final result of someone’s thinking. They don’t often witness what is going on inside someone else’s head to see that they too grapple with and sort out their confusion. A student can control their frustration and confusion by first recognizing that it is a normal part of the process and secondly by identifying what exactly is causing the confusion rather than stating in general terms, “I don’t get it”. Students must learn to use this productively. This is done over time with much encouragement and acceptance. Students need lots of experiences with lots of feedback before they will begin to see the big picture and identify where each piece fits. Like Dubinsky, I think understanding accumulates slowly and comes together to form a connected, cohesive representation.
How my theory works/How it can be used:

(You’ve read much of this before because I turned it in to you earlier for some feedback. I’ve integrated my answers to the questions you raised earlier. They were very insightful! If you’ve got other suggestions/ideas/concerns I would really appreciate it. This has been an excellent experience for me!)

Proficiency in mathematics is necessary for a person’s success both in school and throughout life. Low mathematics achievement is correlated with a number of social problems including high school dropout rates, delinquency, unemployment and homelessness. Regardless of the importance of learning mathematics or the detriment of not learning it, it has become acceptable in American society to claim a lack of ability to learn mathematics to then become excused from learning it. According to Marilyn Burns in her book, *Math: Facing an American Phobia*, “The negative attitudes and beliefs that people hold about mathematics have seriously limited them, both in their daily lives and in their long-term options” (p. ix). The call has gone out for serious work to be done to make mathematics more meaningful and improve the attitudes and beliefs of the general population. Recent research and development on cognitive development theory and informational processing models can assist in the description of the complex mathematical concepts students study as well as assist in the intervention process as well, making progress toward improvement possible.

One mathematical concept that is frequently encountered in everyday life is that of decimal fractions. Decimal fractions are everywhere – on the labels of bottles, cans, or boxes that we use, on tickets or tags where we shop, in advertisements or newspaper reports that we read, on the library shelves where we retrieve books, on gas pumps, on
meters, and on the gages in the cars that we drive. Despite their regular occurrence, the persistence of problems working with decimal fractions has been well-documented (Bell, Swan, & Taylor, 1981). On the fourth National Assessment of Educational Progress (NAEP), basic misconceptions about decimal fractions were held by half of the seventh graders (Kouba, Carpenter, & Swafford, 1989). There is evidence that these misconceptions are lasting and continue into adulthood (Putt, 1995; Thipkong & Davis, 1991; Stacey, Helme, Steinle, Baturo, Irwin, & Bana,?). A major concern lies with those students who progress through the instruction of decimal fractions, beginning with formal instruction in about fourth grade and continuing through middle school, and still do not grasp the concept. This happens all too frequently. Given the importance of decimal fraction concepts, intervention is necessary. Before this can happen, we must clearly understand the problem.

This is not a new idea. As Hiebert states in his article (1984), “If instruction is going to build upon children’s strengths and remediate their weaknesses, we must become aware of how mathematics looks to them.” In order to “prescribe” an appropriate program for students we must consider a description of their learning behavior that is dynamic and allows for continuous change and growth. It must allow a student to enter at any point and move freely as their learning changes what they know.

The concept of decimal fractions is built from prior experiences. Exploring exactly what prior experiences and how important those experiences may be is the purpose of this study. The background knowledge that students bring with them may be procedural, conceptual, or a combination of both. Because of this, the measurement of understanding is not a “have it” or “don’t have it” type of situation. Understanding
develops in stages, which increase in sophistication and complexity over time (Piaget, 1950; Bruner, 1964; Biggs & Collins, 1991). Instruction of decimal fractions occurs during the concrete symbolic stage of development from about age 6 to 16 (cite study, 1995). According to this research, there are “learning levels” within each of the stages of development, including the concrete symbolic stage. These levels indicate the progression of a child’s ability to move from the basic elements of the stage “to an integrated and sophisticated use of those elements”.

This theory of SOLO Taxonomy (Biggs & Collins, 1982) can be applied to decimal fractions. Three of the five levels associated with the SOLO Taxonomy were observed (1995) in a study of decimal fraction development:

1. Unistructural responses, which represent the use of only one relevant aspect of the stage
2. Multistructural responses, which represent several disjoint relevant aspects of the stage that are processed most often sequentially
3. Relational responses, which represent an integrated understanding of the relationships between the different aspects

The construction of the concept of decimal fractions was actually found to be more complex than the understanding of common fractions using this classification system, due to the sophistication of the aspects that need to be tied together. Research indicates that there are actually two unistructural-multistructural-relational (UMR) cycles that exist for the concept of decimal fractions within the concrete symbolic stage of development.
The goal of my study would be to define the many “aspects” of development to better understand student deficits and to inform instruction in a way that moves students more efficiently through the learning cycle that exists for the very complex and sophisticated concept of decimal fractions. Rather than moving from one discrete level to the next as SOLO Taxonomy suggests, the lines can be blurred to allow for the “transition” from unistructural to multistructural to relational. The cycle suggests that students are back where they started and seems to disregard the fact that they have made progress toward a better understanding of decimal fractions. I propose the use of a continuum where specific structures and the relationships between them could be identified in a more overlapping fashion. This would enable the measurement of small shifts in progress.

**Continuum of Benchmarks of Decimal Fraction Readiness and Early Understanding**

The continuum of benchmarks of decimal fraction readiness and early understanding would allow for early identification of students who are at risk for problems working with decimal fractions. The continuum also would allow for a better analysis of where students fall developmentally on the continuum of decimal fraction understanding. Procedures to assess basic decimal readiness are needed to measure performance on necessary background skills and concepts before and during the time children learn to work with decimal fractions until the concept is mastered. The assessment should identify those children who are lacking critical skills and concepts regardless of age, grade, or developmental stage as well as identify the skills and concepts each child is lacking. The assessment should also enable educators to monitor the progress and growth of these skills and concepts over time. The assessment should
help inform instruction of decimal fractions as well as measure the effectiveness of instruction for individual students.

The key words in this assessment are continuum and benchmarks. It is necessary for the assessment to be on a *continuum*. This makes it possible to assess student understandings on an ongoing basis to monitor how students are progressing over time. It has been suggested that learning decimal fractions can take several years as a result of the difficulties students have with the concept (Sackur-Grisvard & Leonard, 1985). The first step is developing this continuum. It is also important that the continuum be marked by benchmarks that indicate important skills and/or concepts that identify progress. These benchmarks should indicate important pieces of prior knowledge that are necessary for the development of the concepts of decimal fractions and the related procedures. They should also have predictive validity when considering future success working with decimal fractions.

Reading and writing development and the assessment of the readiness of them requires a similar assessment procedure. One such program to assess reading readiness is the Dynamic Indicators of Basic Early Literacy Skills. Like DIBELS, the continuum of benchmarks must be, “(a) easy to administer, (b) capable of repeated and frequent administration, and (c) time efficient and cost effective” (Kaminski & Good, 1996, p. 216).

Similar to DIBELS and the indicators related to early literacy skills, decimal readiness appears to be tied with numerous prerequisite skills/concepts. These prerequisite skills/concepts (as identified by research), which have the potential to affect the learning of decimal fractions, will act as benchmarks. They are as follows: (a) place
value, (b) composition of numbers, (c) skill with whole numbers (d) the separation of
decimal fractions from whole number conception (e) representation of numbers, (f)
partitioning, (g) estimation/number sense (h) use of context. A genetic decomposition of
these skills/concepts has been included to illustrate the overlap and connections between
them.

**Guiding Questions:**

1. What are the important prerequisite concepts/skills to learning decimal fractions?
2. How do they flow?
3. How can they be used to inform educators of student “readiness” for learning
decimal fractions and the progress that is made toward “readiness”?

**Purpose:**

The purpose of my study would be to assess readiness to learn decimal fractions
based on identified readiness indicators and to trace changes from intuitive/informal
methods (unistructural responses), to a broader understanding of decimal fractions
(multistructural responses) and finally to a more powerful, integrated understanding
(relational responses). This study will extend the multistructural response level by
identifying necessary aspects of decimal fractions that need to be addressed
independently in order to move to a relational level. The continuum will allow for the
measurement of small shifts in progress between these levels. The large leap from multi-
structural to relational is too big to be informative. How does this progression happen?
This is what I would like to find out. This information will be used to build a model of
student thinking and the defining characteristics at different points along the growth
continuum. If deficient prerequisite skills/concepts can be determined then this
assessment can inform instruction in order to assist students that would otherwise be at risk of inadequately learning decimal fractions.

**Potential methodology:** (This is based on the input you gave me previously. Do you think I’m on the right track?)

We have students in grades 8-12 taking a minimal competency test (MCT), which includes procedural type questions regarding fractions, decimals, and percents as well as skill with whole number, estimation, and place value. I’m considering looking at those to begin with and looking specifically at those questions on the MCT that are related to decimal fractions as well as those related to the indicators (whole numbers, place value, equivalency, etc.). I could look for relationships between questions answered correctly and those that were missed to begin developing a map of how the “benchmarks” I’ve identified are related to each other. I would look to see if there is any information this could provide in the sequencing of the skills or in creating a more accurate genetic decomposition and create an initial continuum.

I would also use this information to identify students at various places along the continuum to interview them for a more detailed picture of their understanding. I would use the number and type of questions answered correctly on the MCT to develop the continuum scale and then list corresponding prerequisite skill/concepts that the student demonstrates at that particular place on the continuum. I would analyze this to see if I can determine an order or priority for these particular prerequisite skills/concepts. These will then become “benchmarks”.

I would then create the Continuum of benchmarks assessment administer it and compare the success with the prerequisite skills to the success with the MCT decimal fraction scores.
fraction items and to the interviews to determine how the prerequisite skills correlate with both the number and types of questions answered correctly on the MCT.

I started a review of relevant literature, but haven’t gotten very far with it yet. I’ve read a number of articles and just included some notes about the skill/concept and people that have done research on that one in particular. If you’ve got other suggestions about how I could go about this I would really appreciate it!

- Place value… (Sowder, 1997-see this article for possible assessment questions; Bell, Swan, & Taylor, 1981)
- Skill with whole numbers - skilled/unskilled
- Separation from whole number conception (Sowder, 1997; Sackur-Grisvard & Leonard, 1985; Resnick, Nesher, Leonard, Magone, Omanson, & Peled, 1989) (Established 3 rules useful as diagnostic tools to detect nature of children’s understanding- use my action research interview questions to assess this); Kieren, 1995; Mack, 1990; Owens & Super, Number and Operations book)
- Composition of numbers… (see Payne, 1984, p.15)
- Representation of numbers… shade models; label numberline (hidden troubles, Payne, 1984, p. 14), (use my action research survey question and number line questions to assess this)

### Meaning of the symbols
- “by the time children reach upper elementary school, the symbol system of fractions and decimals are sufficiently complex that it is not easy to establish meaning for the symbols, even with considerable use of concrete referents” (Hiebert, 1984) shaded parts of a single region (Payne, 1976) & parts on a number
line (Behr, Lesh, Post, & Silver, 1983; Larson, 1980; Payne, 1976) – “Decimals represent the integration of part/whole concepts of fractions and place-value concepts of whole numbers.” (Hiebert, 1984)

- Links between fraction and decimal form (Hiebert, 1984)
- Concept of a fraction/partitioning (Kieren, 1980; Pothier & Swanda, 1983; Hiebert, 1984) … Partitioning is the act of dividing a whole into equal-sized parts.
  “Partitioning process that underlies the decimal representation of number is subject to the added constraint that a whole must be partitioned into exactly 10 equal-sized pieces.” (Hiebert, 1984). “In summary, the evidence suggests that upper-elementary school children have some important foundational understanding of fractions and decimals” (Hiebert, 1984; p. 504)
- Contextual understanding…
- Estimation skills/number sense…
- Use of reference points/benchmarks (0, .5, .25, .75, 1) (McIntosh, Reys & Reys, 1992)

References:


