MY THEORY OF MATHEMATICS

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The purpose of this paper is to outline my personal theory of mathematics. Although I have limited practice in the implementation of my theory, I will lay out many principles in the following that will give a perspective on my vision. After all, a theory is just an opinion which could possibly be true. I have however, supported my opinion through research and sound theories of others.

The understanding I posses of mathematics stems from my experience throughout school. I was a straight A math student. I could memorize anything and ace every test. People actually thought I was this amazing mathematician just because I had a heightened ability to memorize procedures and rules and apply them on tests. My teachers mostly taught what I had to know to pass the test. They never had me reflect on what we were discussing or challenge me to think deeper. This style and theory of understanding mathematics gave me the ability to learn to do problems without knowing how to do them and why that is. In essence, I had little true understanding of Mathematics.

It wasn't until I came to college and tried to pass courses such as Calculus and Foundations of Geometry that I realized there was more to math than just memorization. I learned more about truly understanding math in four years than I ever had in my previous math experience. Why is that? What changes between that time frame that makes the gears switch from regurgitation of facts to understanding how these facts and laws came about?

It is clear to me that understanding is more than memorization of facts and procedures, even though I see this cycle repeating itself everyday. So many of the students are so busy trying to remember the rule for this and the rule for that, that they have lost any understanding in the whole process of it all. There is a place for some memorization in mathematics. All individuals, unless hindered by some disability, need to know basic addition, subtraction, multiplication, and division facts. For those who do
not have this ability, they need to be taught how to use tools, such as a multiplication chart, to help them overcome this deficit. Students also need to memorize terminology and the basic language of mathematics. When you choose a book to read, do you choose a book that has many unfamiliar terms or do you choose a book that you can mostly understand? The same is true for math. Students have to be able to understand mathematical language. They have to know what symbols such as +, x, /, <, and = mean so they can create and interpret number sentences and gain a true appreciation for the grammar and language of mathematics.

Once students develop this basic knowledge through memorizing, they must be able to comprehend and apply this information in more complex and difficult problems. If students want to be able to understand concepts such as adding and subtracting decimals, they will be unsuccessful unless they are able to apply the basics of what they know about adding whole numbers. For this, they also have to have an understanding of place value and what a decimal point means.

Understanding also means that students are able to apply the knowledge they have gained in new situations and contexts. Students who understand can do not only written calculation problems, but can also do problems where they have to infer information. For example, students who understand Calculus can use what they know about finding a derivative and apply it to finding the slope of a line, and interpret what this means. This learning multiple contexts and situations can help lead to higher-order thinking skills. Understanding develops through consistent reflection and practice. When individuals are asked reflect or interpret what something means, they are challenged to dive into analysis and evaluation, both higher-order thinking skills.

The level at which individuals are then able to control all of their strategies and tools for learning also establishes their true understanding. Schoenfeld stressed the importance of control in his work, and I agree with this notion. Deeper understanding requires that the correct learning tools are picked out in an efficient manner. Students who are able to learn skills in properly utilizing all of the tools tend to find more success. I often think of our head as a library. Students who are struggling with understanding tend to have the old card catalog system of pulling out the drawers and searching until
you eventually find it. Students that have a deep understanding have the computerized card catalog. They are able to type in what they are needing and come up with a few options. From there they are able to decide which of these options work best.

I feel that the final component to understanding involves which assessment techniques one is successful at. Students that possess memorization skills with little understanding tend to do well at multiple choice, computational, and convergent testing questions because there is a straight path to answer, with little thought involved. Student possessing understanding can excel at open-ended, divergent testing. They are able to successfully find solutions and explain why and how they used the strategy to obtain a solution to the problem.

This reasoning is why I feel many students struggle at understanding mathematics. They lack the skills to put their knowledge into words and synthesize their information abstract situations. Mathematics is one of the hardest subject areas to express because it does not have the similarities to English that most other subjects do.

Students successful at understanding have control, basic knowledge, the ability to reflect on what they are learning, the ability to transfer knowledge, and the ability to apply it in other situations with different contexts and situations with heightened complexity and difficulty.

I have made it clear as to what it means to understand and to not understand mathematics, but I have not yet made it clear how mathematical understanding develops. I have given hints of how mathematical understanding develops throughout the beginning of this theory, but would like to take to opportunity to solidify them now.

Bloom’s Taxonomy and the ideas of Fruedenthal and Van Hiele play a large role in the basis of how understanding develops. Understanding does develop from the basic level of knowledge and then proceeds on the final processes of synthesizing and evaluating. Individuals begin with the basic knowledge way back in kindergarten and first grade. They begin with mathematical processes that must be memorized such as counting, adding, and subtracting. They begin to develop a basic understanding of Algebra as a repetition of patterns. As they progress through their elementary years,
this knowledge needs to progress into comprehension, or understanding why and how to use their basic knowledge in more complex situations. I am not convinced on Van Hiele's idea of a quantum leap. I think that it might be possible, but unlikely. For students to understand deeply they must be able to achieve higher order thinking skills such as analyzing and evaluating. Through these higher levels of thinking, students are better able to distinguish which schema they already possess that would fit into the subject matter. For example, I might have schema for what decimals are and what fractions are, but when I am asked to distinguish how these are similar and different my schema is able to accommodate more information about these topics.

The beginnings of understanding and building schema also require skill for learning or the achievement of a level of learning. There is a tremendous difference between the understanding of a novice and the understanding of an expert. Novices understand mathematics at a very basic level and struggle with the analysis and evaluation process. Experts on the other hand, can analyze and evaluate consistently with ease. Both understand mathematics, but novices have not yet had the time and experiences to develop into experts.

Allotting sufficient time is very important for comprehension. Understanding takes time and continued practice. It is not good practice to give students the same problems over and over. Good practice involves increasing both the complexity and difficulty as you practice. Giving students problems in different contexts and forms also increases their understanding. It however does take time to implement good practice. Students will not retain information as efficiently if they are just spoon fed it. If they are given time to explore and investigate with some instruction, they will be more successful.

The investigations do need to be rich and deep, but must allow students to ease into it. For example, students will be unable to develop understanding for a multi-step word problem until they understand how to solve each of the sub-components. You have to build individuals for success by teaching them smaller parts and skills before looking at the big picture. Instructing in this manner will insure that successful understanding can follow.
One way to develop understanding is through motivation. It is similar to the idea that a nicotine patch will not help a smoker to quit smoking until they truly have the desire to quit. Learning and understanding is the same analogy. Individuals will not put the effort into learning unless they have a motivation or desire, whether intrinsic or extrinsic. Many people at lower levels require extrinsic motivations such as “games”, praise, rewards, good grades, and competition. Individuals at higher levels often desire intrinsic motivators such as desire to learn more, wanting to improve knowledge, and the ability to reward oneself. Regardless of the type of motivators, these motivators must be sparked and developed in some way.

One means of keeping motivation high is to present material in a variety of ways. Howard Gardner has spoken of this theory of multiple intelligences and describes how this also helps develop better understanding (Gardner, *Frames of Mind: The Theory of Multiple Intelligences*, 1983) Each one of us has one or two areas that we tend to learn better in, whether it is kinesthetic or logical. Presenting information visually, kinesthetically, musically, and verbally can enhance development.

Development is further enhanced also through transfer. People have the amazing ability to transfer knowledge between contexts, subjects, and situations. One of the best ways to develop transfer to enhance understanding is to make individuals aware of what strategies they are using in their learning. When individuals actively monitor their learning process, they become more successful at developing understanding (National Academy Press, *How People Learn*). However, this monitoring also requires consistent, appropriate feedback from the facilitator. When monitored appropriately, the transfer of these positive strategies enables successful development.

Further enhancement of transfer can be accomplished by using contrast. For example, students have knowledge about what fractions look like, but it becomes clearer when you use a Frayer Model Graphic Organizer to contrast examples of fractions and nonexamples of fractions to make the concept clearer. From this facts and a definition can develop.

The Frayer Model is just one example of a graphic organizer that develops understanding. Graphic organizers are powerful tool and strategy to use to further
understand information. Other great organizers that increasing understanding are the Kite method of problem solving using characteristic organization to discover likes and differences of terms. In this method, students are asked to identify what the problem is asking. Then they identify what they know and need to know in the problem. They devise a plan through words or diagrams and then solve the problem using their strategy. The last component asked students to then reflect on their answer and strategy.

So far, I have touched on the importance of time, practice, transfer, and the use of graphic organizers to develop understanding, but I would like to take the time to stress the importance of preexisting knowledge and schema in the development of understanding. One way to encourage individuals to begin thinking in the right context is to discover the prior knowledge they have through questioning, predicting, or KWL’s. This prior knowledge and schema that students bring with them might not always be accurate, so it is important that you are aware of this so that you know how to address this. It is just as important to address misconceptions students have as it is to address the concepts. Skemp asserts that students have to have some type of crisis to change this schema, and I am not convinced that this is true. I think that it is more of a process of replacing old ideas with newer, more complex ideas through a gradual process. Individuals need multiple experiences in the correct situation to overcome their misconceptions in their preexisting schema. After receiving these multiple experiences, they still might not have replaced their old schema, but they keep these new schema stored. Eventually, students will have an “Aha Moment” where they suddenly fully understand the new schema and now discard the old schema. It might happen during a completely different experience. For example, students might have schema for the adding of fractions. They might believe that you just add the numerator and denominator straight across. I would show them pictoral models, use pattern block and Cuisenaire rods to change this misconception, but students might not have this “Aha Moment” to change their previous schema until we begin multiplying fractions. It is
often challenging to overcome misconceptions, but these views must be extinguished for understanding to occur.

Since students bring their own schema and experiences with them, it is important to relate information back to everyday life. Fruedenthal addressed the importance of schema in his works, and I agree with many of the assertions that he made. One example of how you can build schema and relate mathematical concepts to everyday life is with equivalent fractions. Students often think that $\frac{3}{4}$ is smaller than $\frac{6}{8}$, even though they are equal. One way to relate this to everyday life example is to relate this to the following:

Dog -> Puppy         Cat -> Kitten

A dog and a puppy are equivalents. Questions students, “Do they look the exact same? What do they have in common? What is different?” Explain to them that they are both dogs. A dog, like the digits in the fraction $\frac{6}{8}$ are larger, but it is equivalent to a puppy, just like $\frac{3}{4}$ is equivalent to $\frac{6}{8}$. You can then proceed in the same fashion with cat and kitten.

Not only should you establish relevance to everyday life, but it is also important to connect understanding to other subject areas, such as language arts, social sciences, and home economics. Individuals can use their experiences in these areas to transfer knowledge and schema to mathematical concepts. Science and home economics are especially full of developmental concepts, although other subjects contain them as well.

With these connections, development also requires a mixture of abstract and concrete mathematical concepts. It is important to start with concrete concepts that can be modeled with objects, such as adding and subtracting positive number before you move into adding and subtracting negative numbers. Even with this you can still use number lines and the relationship to temperature to develop this understanding, but it is not as clear as using positive numbers. It is much more abstract because we do not have very many negative things in our world.

The final and very critical point I would like to address in regards to how understanding develops is to talk about the role of reflection in enhancing understanding. It is amazing how much individuals are able to think and self-assess
when they are able to reflect. This reflection should involve analyzing what one knows about the subject matter, and what they have learned and understood so far. It should include questions that reflect concepts or topics they are still struggling with along with topic they still want to explore. Reflection can definitely be the linking tool, to all of the strategies above mentioned, in developing understanding.

I have outlined my theory giving a few examples to clarify, but I have not yet shown how my theory works in practice. To demonstrate how this works, I would like to apply my theory to learning fractions. Before even beginning to teach this concept, students must have knowledge of what whole numbers are and be able to give an example. They also need a background of factors and multiples. When first beginning any unit or topic, I informally assess prior knowledge through an organized process, such as a KWL. In a KWL, students are first asked to tell me everything they know about the topic (K). Then through discussion and sharing, they decide what they want to know (W). After we finishing covering the topic, students are asked to reflect on what they have learned (L). I also often times asked students to add a last component to this method, application. Students are asked to apply how this knowledge is useful and apply it to other contexts. Think-pair-share is a wonderful strategy to help spark students thinking, so I often use this strategy to engage students. I have students tell what they already know about the topic, so that I know what skills they are bringing, and get a feel for the range of my students’ levels.

I also find out what students are wanting to know. Often times, this is more difficult, but it is important to make students feel responsible for their own understanding. Again, think-pair-share, along with questioning techniques, can really get discussion going on this.

Once prior knowledge is established, and a focus is chosen for what students want to know, it is time to begin. First of all students need to know what a fraction is. It is important to use some example, such as the concept of a whole number, to contrast with what a fraction is. It is important to stress to students that you are worrying about names, not the changing of numbers. Using the Frayer model, shown below, is one
example of comparing these concepts. As you notice, I list 8/1 as an example of a fraction. 8/1 has the same value of 8 and is equivalent. For naming purposes, 8/1 is written as a fraction and 8 is written as a whole number.

<table>
<thead>
<tr>
<th>Whole Numbers</th>
<th>Fractions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Definition</td>
<td>Definition</td>
</tr>
<tr>
<td>A number zero or greater that in the sequence 0, 1, 2, 3, 4, ...</td>
<td>A number composed of two or more digits in the form a/b.</td>
</tr>
<tr>
<td>Facts</td>
<td>Facts</td>
</tr>
<tr>
<td>-Zero or greater</td>
<td>-are composed of two or more digits</td>
</tr>
<tr>
<td>-Can’t be %, ., or fractions</td>
<td>-can’t be a %</td>
</tr>
<tr>
<td>Examples</td>
<td>Examples</td>
</tr>
<tr>
<td>7 346 1</td>
<td>5/6 5% 2.798</td>
</tr>
<tr>
<td></td>
<td>56 2.39 2%</td>
</tr>
</tbody>
</table>

Once this notion of fraction is established, the first topic to address is how to find equivalent fractions. Concrete examples must be established before abstract concepts can be accomplished. Methods of developing the notion of equivalent fractions involve using manipulatives, such as pattern blocks, Cuisenaire rods, fraction circles, and fraction strips. With these, students are able to manipulate, and with guidance, discover valuable concepts of equivalents. To expand on this and create a more abstract concept, I explained to students that to make a fraction in simplest form you have to find an equivalent to the fractions, with the two smallest digits. In other words, the only factor left that divides both of the digits is one. The idea of simplifying fractions is very tricky for students because they often see it as reducing a fraction to a smaller fraction even though you are actually dividing by another name for one.

After these concepts are mostly understood, I move onto comparing fractions to extend the concept of equivalents. Students are actually putting the idea of equivalents in a different context. For example, I play games with them. I ask a student, “I have a Hershey’s candy bar. I give Lucy 2/4 of the candy bar and you 3/12. Is that fair? Why not?” Students have to take comparing one step further into comprehension, and the units the candy bar is divided into serves as a concrete model.

What really forces students to clearly understand these earlier concepts so that they can apply them is operations with fractions. Addition and subtraction often give students the most difficulty. I use manipulatives with addition and subtraction to
reinforce the concepts. I also include real life example associated with cooking and measurement with rulers to help students in these concepts. I find that if they first see what is happening visually and kinesthetically as they are performing the operations, the process is understood much easier. Although subtraction and addition are more difficult for students, multiplication is easy for most all students to memorize the process. Unfortunately, most students do not truly understand why this is. I develop this understanding through use of the Area model of multiplying fractions.

Dividing fractions is by far the most procedurally taught portion of fractions. It is important develop an understanding for why multiplying the first term by the inverse of the second is the same. Unfortunately, this is a very abstract concept and is hard for most students to grasp. The method I have found success in is using diagrams to find the answer and then using this to prove that this process works. In essence, we are solving it through use of a proof.

I think that from this description you are able to see how my theory works in practice. I can continue on about developing higher order thinking skills by connecting fractions to decimals and percents. I think that by doing this, I would only cause repetition, since the purpose and practice of my theory has already been established.

To further research and develop my theory, I would like to study and research the learning of fractions more in depth. I am curious to see whether or not it would be easier for students to understand operation of fractions if multiplication and division were taught and discovered before you taught them how to add and subtract. I can see how this approach, in theory, could be successful. After all, when you are finding equivalent fractions, you are essentially just multiplying by another name for one, such as $2/2$. When you are simplifying fractions, you are just dividing by another name for one, such as $4/4$.

I would love to take the opportunity to experiment with this idea. To do this, I would need two classes with similar needs and range of levels. I would also need to complete research to see if similar studies have been done that could also apply to this. I
also want to interview students before, during, and after the process to accumulate various forms of assessment.

I feel that one thing that could further my research the most is to talk to other teachers to get their views on it, and what they have tried before. This might help me to understand what implications teaching in each manner would have on further learning. Do we teach adding and subtracting of fractions first just because it is that way in the textbook, or is there research to prove this is how students best understand?

My theory of understanding mathematics suggests that it could possibly set students up for greater understanding if they learned how to multiply and divide fractions before learning how to add and subtract fractions. I hope to research this topic further in the next two years of my teaching experience.