My Current Understanding
of
Understanding Mathematics
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When I think of the understanding of mathematics, I think in part of a state of mental awareness, a level of comprehension reached due to the arrangement of particular cognitive structures, whereby one can access and utilize knowledge of propositions and algorithms pertinent to solving appropriate tasks and problems. In this sense, I see understanding mathematics as a level reached after certain prerequisite knowledge has been accumulated. It is a part of a continuum of learning (figure 1). On one end is rote knowledge, on the other, deep understanding, with many gradients in between. I would differ with Van Hiele (Van Hiele, 1986) in this aspect: I don’t think the levels can be as neatly delineated as the Van Hiele levels, rather I believe it is possible to locate a person further towards one end or another relative to previous understanding, or in comparison with another person at a noticeably different level.

![figure 1: A Continuum of Learning](image)
Certain tasks do not require the progression of learning beyond the rote knowledge level (for example, problems such as adding whole numbers), but there are many that require the math student to move further towards the deep understanding end of the continuum. As students progress in their studies of mathematics into higher-level math courses, the need to move into deeper levels of understanding becomes imperative.

I would add, however, that understanding mathematics does not necessarily follow from knowledge gained. This is apparent from observations of students who can memorize and apply algorithms with little or no understanding of the mathematics behind them, but are able to do so only in limited situations.

Understanding requires the ability to access some subset of memory containing relevant knowledge, which means of course that this knowledge must be present in the first place. At some point in time previous to understanding, mathematical facts must have been stored in order to give some basis of knowledge on which understanding can act. The way I see it, knowledge of facts is a prerequisite for understanding, but understanding is not guaranteed just by the presence of knowledge in the form of accumulated facts.

Anderson, Reder, and Simon lament how modern mathematics education seems to be trying to reject the “essentialist view of mathematical knowledge” concerning accumulated facts. My personal view is that accumulated facts are important, being a necessary precursor to understanding. They should not be viewed with distaste or rejection. At some elementary level of learning any given topic, they are vital to the student. They are not, however, the be-all and end-all in mathematics, and students should be encouraged to move beyond mere memorization whenever feasible.

I think most people would agree that understanding obviously goes beyond the mere accumulation of facts. It involves discovering for oneself patterns and connections, and gaining for oneself the insight that enables one to see objects, problems, and situations in ways that promote creative mathematical thinking directed towards specific applications and solutions. Understanding enables one to seek alternatives, and evaluate and apply the best alternative.

Understanding enables a focusing of abilities and knowledge relevant to application towards the situation at hand (usually for the student this is in some form of a problem). I would tie this in with Schoenfeld’s ideas on “control” (Schoenfeld, 1985) and state that control results
from understanding. Furthermore, increased control goes hand-in-hand with increased understanding.

**Understanding enables one to learn from failures, and evaluate performance, thus leading to an even deeper level of understanding.** Pólya’s last step to problem solving, “Looking Back” (Pólya, 1957), touches on this evaluation step as being important in “consolidating knowledge”.

Understanding allows students to discern relevant information, apply knowledge thoughtfully and productively, interpret novel situations, and see mathematical facts as part of a conceptual whole, a greater body of knowledge. Students who merely memorize without understanding do not achieve this level of capability. They see memorized facts as separate units and are unable to apply these little chunks of information towards solving novel problems that extend beyond examples they have seen or memorized. They cannot see the bigger picture and therefore cannot make the leap to generalization and abstraction, which is the very strength of mathematics.

How does understanding of mathematics develop in the first place? I believe that understanding is a function of *time, effort, experience* and *knowledge*. It generally takes some amount of *time* and *effort* studying mathematical concepts in order to obtain useful and applicable understanding of the concepts. As I mentioned before, the *knowledge* must be there in the first place to act as a basis for the understanding. *Experience* comes into play in the sense that in order to understand mathematics, one must experience it personally through study and contemplation.

It is this self-experience (based on our own contemplative efforts and the self-discoveries that result from them) that enables us to build the proper cognitive structures needed to achieve understanding. From a somewhat constructivist point of view, I believe that understanding is to be found in the individual’s internal state. I also believe that understanding of the same concepts can differ between individuals (understanding is unique to the individual at some level), and that these different perspectives can still lead to successful interpretations and solutions to mathematical problems.

I greatly value the concept of reflective abstraction as put forth by Piaget and developed by Dubinsky. Reflective abstraction is, I believe, a *primary* mechanism for achieving progression of learning from merely knowing facts to the level of understanding concepts. This reflection, contemplation, and expending of mental effort studying mathematics is something
that can be done only on a personal level. I, as a teacher, cannot make my students understand, nor can I understand for them (although I can structure my instruction so as to increase their chances of understanding). Understanding is a level that my students must arrive at, in the end, for themselves, after expending effort (occasionally great effort).

My current beliefs about understanding fit in well with the concept of schema. My thought is that understanding occurs when certain cognitive arrangements take place due to interactions between knowledge, experience, and effort in the form of study and reflective abstraction, over time. Once these arrangements are achieved and comprehension takes place, all subsequent schema (actions, objects, and processes forming a coherent structure) connected to them are affected. I believe that schema will change to accommodate new situations (Skemp, 1987), and that schema used by students in order to make sense out of problems vary between students, and even within individual students, depending upon the type of mathematical task they are carrying out (Dubinsky, 1991).

Barring physiological damage, understanding is not lost. It may, however, be modified by new information and experience. My thought is that deeper levels of understanding can cause previous concepts to be modified or changed entirely, so that the process of moving towards the right on my learning continuum can affect cognitive structures gained previously from moving through levels left of current levels of understanding. In fact, greater depth of understanding may even cause one to realize that the understanding one had at a previous time or on an earlier concept was faulty. This is another belief I currently hold: that understanding can be correct, incorrect, or incomplete.

Understanding is not a static thing. It is constantly changing with every new bit of information entering the brain, every new experience. It can occur gradually over time, or take place in rapid leaps of insight. It may not occur at all. I believe that, at the smallest level, concept changes necessary for understanding take place in quantum jumps (after all, electrical impulses behave that way, and that is how our brain operates). These quantum changes can occur, bringing about a reconstruction of schema that can allow understanding to follow.

What causes this restructuring of schema that allows understanding to occur? At some point in the student’s experience, some obstacle to further understanding must arise. As Dubinsky puts it, when the student is not successful in employing current schema to a task, “her or his existing schemas may be accommodated to handle the new phenomenon.”
(Dubinsky, 1991). My feeling is that at some point, the students realize that their current understanding of a situation is not sufficient to enable a successful interpretation and eventual solution to a problem. A cognitive crisis occurs. Assuming that they don’t just give up but instead look deeper into the problem, reflecting on everything they know relative to the situation (similar past experience, pertinent facts, current understanding, etc.), experimenting, evaluating and learning from the results, they may eventually experience that restructuring of cognitive processes necessary to lead to deeper understanding. There is, of course, no guarantee that this will occur, and as Skemp points out (Skemp, 1987), old schema may present such an obstacle that further learning does not take place.

Earlier I mentioned that I view understanding as part of a continuum of learning, with rote knowledge and deep understanding on opposite ends. I would like to relate this to the development of understanding by observing the fact that one usually doesn’t jump from one end to the other (rote knowledge to deep understanding) without crossing through some gray area in between. If it does happen, I would say it does so rarely. At least, that has been my experience both personally and with students.

Let me give a specific example of how my current beliefs on understanding fit into actual classroom situations. Let us look at solving quadratic equations. Students are introduced to this concept in an algebra course (usually Algebra I). They have previous knowledge of basic math operations on real numbers (addition, subtraction, multiplication, and division). They learn how to take an equation of the form \( ax^2 + bx + c = 0 \), and, using algorithms of factoring, completing the square, or the quadratic formula (based mostly upon rote memorization at this point), they learn how to solve for \( x \). They may develop some sense of understanding at this point based on recognition of certain types of equations requiring certain algorithms for solutions, or by seeing factoring as a “reverse FOIL method” based on previous experience in multiplying binomials.

At this point, there arises a need for students to expand their understanding of quadratic equations to solve problems of greater complexity, beyond simply being presented with a quadratic equation directly and being asked to solve it. Current understanding is not sufficient to do this. Cognitive crisis occurs, and schemas are restructured.

Next, students may be introduced to the concept that a quadratic equation is a particular type of a quadratic function of form \( f(x) = ax^2 + bx + c \), namely, one where \( f(x) = 0 \). At this point in time, they will have gained basic knowledge of functions in general and, ideally,
developed some understanding about functions as a result. This would include knowledge about graphing functions.

Upon further study and after some time has passed (during which they have gained personal experience by practicing solving problems), the students see examples of, and begin to reflect upon, the graph of the quadratic function as being that of a parabola. They begin to understand that when they are solving a quadratic equation and setting \( f(x) = 0 \), they are in fact finding the values of \( x \) where the graph crosses the \( x \)-axis (where \( y = 0 \)). They expand the situations where they can apply their knowledge through solving problems of greater complexity, including story problems. They study the discriminant, and make the connection between the sign of the discriminant and the types of solutions the quadratic equation has. They tie this in with their understanding of the graph of the parabola and where, if ever, it touches the \( x \)-axis. Furthermore, they reach a point where they are able to generalize concepts of solving for zeros of functions beyond quadratic functions, and on to higher order polynomial functions.

The isolated bits of knowledge gathered up to this time have become more generalized, more abstract, leading to a deeper (certainly more than a rote memorized) understanding of what quadratic equations are. The students started on the rote knowledge end of the continuum, and, through time, experience, and effort in the form of reflective abstraction, they have moved closer towards the deep understanding end of the continuum.

How do my ideas fit into instruction in general? Instruction in the classroom should be geared towards helping students move past rote knowledge to deeper levels of understanding. This should not, however, occur as a result of ignoring the importance of memorization of mathematical facts at some level. Remember, those tools of knowledge must be present to be used in the building of understanding.

I believe, to a certain extent, a progression of understanding is built into the sequence of math courses taught. It is intrinsic to the nature of mathematics, as it has developed over the years. Full advantage should be made of this natural progression.

Students begin school learning basic rules, memorizing multiplication tables, etc. They progress to algebra and geometry courses and are then exposed to concepts of greater abstraction than previously encountered. This continues through higher-level courses in mathematics.

Within a given course, according to my current theory, students should start a topic by covering mathematical facts concerning that topic and form that knowledge base. In algebra, for
example, they should review properties of real numbers, such as properties of equality. They should experience this knowledge as it applies to solving simple equations.

Equations explored in examples and study exercises should get more complex over time (linear, quadratic, exponential, etc.). Experience should increase. Word problems should be studied so that students develop further understanding, in order to pick out pertinent information, and in order to set up equations. Related concepts should be investigated so that students can see different aspects and applications of the mathematics involved.

As put forth in NCTM’s Principles and Standards for School Mathematics, “In order to find a solution, students must draw on their knowledge, and through this process, they will often develop new mathematical understandings” (NCTM, 2000). The problem solving aspect, as discussed in the standards, is of primary importance. It is through contemplation of problems and experience working with them that understanding can develop (effort through study and reflective abstraction). Teachers should make sure to include problems sufficiently challenging to students so that the students’ abilities can be stretched. Greater effort should be required.

Another important aspect of my theory is time, and unfortunately I see this as a major problem area in the classroom setting. In our current education system, with the stress on standardized testing and “No Child Left Behind”, the tendency is fostered to cover material so that enough mathematical facts are known to ensure acceptable test scores (breadth without depth). With this philosophy, and the size of many classes, time is not available to foster what I would consider acceptable levels of understanding in a majority of students. I consider an example proving this to be the fact that many students entering Algebra II do so without solid understanding of Algebra I (or even Pre-Algebra) concepts.

Also, related to time, students learn at different rates, and will reach levels of understanding in different amounts of time. It is wrong to think that a class consisting of so many different students of different abilities will, in a set amount of time, have every student functioning at some given level of understanding.

Throughout the instruction process, some form of assessment of understanding (oral or in writing) needs to take place to evaluate a student’s location on the learning continuum. Instruction should then be modified in light of evidence that more practice is needed with basic concepts, or that certain types of problems should be stressed, or that remediation of basic facts is needed, etc. Whatever could reasonably be done to move the student to a greater depth of
understanding (farther along the continuum towards the right) should then be done. I, personally, would like to explore the role of motivation in developing mathematical understanding. My feeling is that the differences exhibited between students motivated intrinsically and those with extrinsic motivations (such as grades) play an important role in the understanding process through the time and effort factors. To what extent this occurs, and how it can be manipulated, could be topics of interest for further research.

References


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