I. What it Means to Understand Mathematics

To understand mathematics is to have the ability to take a contextualized problem, assimilate or adapt an existing schema to the problem, and effectively apply control to the problem solving process in order to find a solution to the problem.

The existence of mathematics seemingly arises from the need to solve problems. In order to solve them, it is necessary to learn certain mathematical concepts and procedures. For students who do not naturally find themselves drawn towards mathematics for its own sake, this approach may provide the motivation for learning mathematics that is desired. The circumstances in which one would use the concepts or procedures are evident. For those who are naturally drawn to mathematics, this approach works well also, as learning control in problem solving and in forming schema are by-products of the approach.

I would describe learning as a linear, continuous, increasing function on intervals, the endpoints of which are points of small jump discontinuities. I feel that as a student learns, he is gradually becoming more comfortable with his learning environment. The initial confusion encountered with the introduction of a new concept gradually lessens while the concept becomes more familiar to the student. Thus, learning grows at a more or less constant rate within these intervals. The jump discontinuities at the endpoints of the intervals represent those “light-bulb moments” as all who are or have been teachers have evidenced that students have when suddenly he/she gets the point of a concept with which he/she has been struggling to learn.
II. How This Understanding Develops

As mentioned above, this process involves an overall context of the problem-solving situation. Within this process, however, are applied several ideas from prominent mathematics researchers. In particular, Van Hiele levels are adhered to most closely, other than the given umbrella of problem solving. The problem solving process could be used in two different ways. The first is in the teaching of a new concept, and the second in actual problem solving by the student.

Before the teaching and learning of a particular concept occurs, the teacher should ensure that students have the appropriate level of initial knowledge and understanding. This would be indicative of Van Hiele’s first stage.

The process should then begin with a problem in which a mathematical concept and/or procedure is introduced, in order to motivate the necessity of learning the concept/procedure. The overall structure proposed for problem solving is an adaptation of Polya’s and Shoenfeld’s problem solving model. A diagram describing this adaptation is as follows.

![Diagram]

Note that the arrows in the diagram go both ways, to and from designing a plan, exploring, and implementing a plan. It is in this circular process that control in finding a solution occurs. As the problem solver works back and forth between these steps, he/she determines whether to proceed with a plan, keep part of it but rework the plan, or throw it out and try a
different strategy. The teacher should lead in modeling control, certainly whenever introducing a new concept in this way, but also when modeling the problem solving process. It is also in this circular process that schema and strategies are found and developed, so that learning takes place.

The initial step in the problem solving process has to do with understanding the problem. This analysis can be done in a teacher-led classroom discussion or in small discussion groups. Students define the problem in their own words, determining given conditions and discussing what they should be finding in order to answer the problem. The teacher’s role should be of coaxing the students to determine these things with appropriate questioning techniques that lead students to find what they could have found themselves. (Polya)

Students should then discuss the problem, designing strategies, exploring possibilities, and perhaps implementing them, depending on the given problem and their prior knowledge. As the problem solving process is used over time, an increasing level of detailed strategies, as Schoenfeld suggested, should be used so that students have many to choose from, and so that they begin to build a variety of schema in order to solve problems. Thus, students are analyzing, designing a plan, exploring, and perhaps implementing their plans.

As students reach the end of what they can do on the problem given their initial knowledge base, the teacher should begin classroom discussion over the groups’ strategies, their successes, and their shortcomings. These discussions over time will lead to the development of students’ control in problem solving. Do the strategies lead to reasonable results? Is there a better way to approach the problem?

The teacher then directs students, through Van Heile levels, in going back to design a new plan that will incorporate the concepts and procedures necessary in solving the problem. In beginning this process, the teacher proceeds to Van Heile’s second stage, guided orientation. The
teacher should direct group activities that lead students to eventually forming an informal set of relations, patterns, or rules that seem to be true in context of the given problem.

The groups can then be brought to a class discussion in which the relations found are verbalized (Van Hiele’s stage three). Students should next be directed to test these relations to see if they seem to work always, or perhaps to find other relations. This corresponds to Van Hiele’s stage four, free orientation.

At this stage, students should go ahead and design and implement a new plan, and continue to solve the problem, given their newly found relations. They should then look back, as they would when using Polya’s model, to determine if their answer seems reasonable, and to see if there would be a better way to solve the problem.

Once this has taken place, students should be brought back together, and led in the integration stage (Van Hiele’s stage five). Students should be taught the contextualized abstraction that has come about naturally through the problem solving process. Here, logical rules and definitions can be formally given. As this process is used over time, abstraction should increase appropriately in order to deepen mathematical understanding. The teacher should give good examples displaying the mathematical concepts and processes, but also should allow students in groups to practice these together to ensure themselves of their ability to perform the processes and understand the concepts.

In assigning student work, skill problems should be assigned so the student has opportunity to perform the skills without complications of a context. However, the assignment should also contain contextualized problems within which the student can find motivation for learning the skills and also practice of the problem solving process. It is during these practice sessions with the problem solving process that the student can begin to build schemas and
control in the problem solving process, on both of which his mathematical understanding can be
grounded.

III. Comparison to Other Theories

I believe that my theory upholds several things concerning learning and transfer from
How People Learn. I have shown agreement with the theory that students need an adequate level
of initial learning with tasks at the proper level of understanding, that they need to have multiple
contexts relating to daily life, and that there needs to be a balance between abstraction and
contextualization.

Obviously, I feel that problem solving is important in understanding mathematics, as I
feel that true understanding is evidenced in the ability to use this understanding to solve
problems. I like what Polya had to say concerning the teacher’s role, that he/she should help, but
not too much. I agree that a teacher’s questioning should lead a student to what he/she may have
been able to come up with themselves, that problem solving should be at the student’s own level,
and that the teacher should model problem solving.

I have to agree with Schoenfeld, however, that more detailed strategies than the ones
Polya suggested need to be formed. I also feel that the problem solving model should indicate
what takes place in real problem solving, and that is going back and forth between designing the
plan, carrying it out, and modifying the plan, as the student sees that a plan needs to be altered or
thrown out. I feel that the desired effective control is an over-riding factor in the ability to
understand mathematics, and that in fact, its absence negatively affects the ability to problem
solve.

I would have to agree with Skemp’s statements regarding the success of adaptability
based on understanding versus understanding based on a rule. I have also incorporated the
concepts of schema, assimilation, and accommodation as put forward by Skemp and Dubinsky. In fact, I believe that my method incorporates schema into a context and into a process by which its formation occurs naturally and is in fact embedded into the method.

As you can see, I appreciate the Van Hiele levels and can readily see their application in the teaching of mathematics. These levels seem to be a natural progression of the learning process. As with schema, in my method, these levels have been embedded into a context and into the problem solving process.

I did not fully appreciate Dubinsky’s genetic decomposition. I found it to be too detailed to actually have the time to prepare a genetic decomposition for every concept or process taught. Therefore, I considered it to be too impractical to use as a method of learning in my classroom. However, I can see its potential use in diagnostics, to find the missing knowledge or understanding in a student who is struggling in mathematics. I can see this being used in a tutorial setting on an individual basis to help a struggling student, as suggested in his writings concerning artificial intelligence.

IV. Examples:

Slope of the tangent line of a function at a point (or derivative at a point)

First, review students on prior knowledge (first stage). They need to know how to find the slope of a line, the meaning of the slope in terms of rate of change, definitions of a secant line and a tangent line, how to evaluate functions, find compositions of functions, and limits.

Give the class the problem. We want to find the rate of change of distance over time (velocity) of a ball thrown straight up in the air, three seconds after it has been thrown. The position function for the height of the ball in \( t \) seconds is given by \( s(t) = -4.9t^2 + 25t + 0.5 \).
Begin the problem solving process. Group students and give each group a function and a point at which we desire to know the slope of the tangent line. Allow time for discussion and exploration. Encourage students to find a strategy to approximate the slope of the tangent line. They may choose to approximate the slope by sketching the tangent line and estimating a second point on the line, and thereafter using the slope formula. Alternatively, they may pick a point on the function close to \((x,y)\) and find the slope of the secant line of the function through the points. By this time, students have analyzed the problem, designed a plan, explored, and implemented a solution. They have thus practiced assimilation of an existing schema, and may have practiced some control in finding this schema.

When these discussions and explorations have taken place, bring the group back together, and teach students, using good questioning techniques, to find the formula for the slope of the tangent line using limits. We go back now to the exploration stage. How can we find a more accurate answer to the problem?

For the second stage of Van Hiele’s levels, have group activities prepared in advance. Each group should use the limit definition to find the slope of the tangent line for a variety of monomial functions. In particular, these should include constant, linear, quadratic, cubic, and radical functions, constant multiples of monomial functions, and sine or cosine functions. Groups should be encouraged to formulate relations between the slope formulas in families of functions.

To begin the third stage, bring students together to discuss results from the group activities. Encourage them to verbalize and formulate rules governing the slope formulas for families of functions. Enter the fourth stage by sending students back to their groups to test these rules for various functions, checking their results by again finding slope formulas by the limit.
definition. Thus, students are again practicing control. Do these rules actually work? Can I use them to answer the problem? Students should also at this time return to their original problem, form a new plan, and complete the problem solving process, finding the slope of the tangent line using the newly formed rules.

Bring students together again for the fifth stage. Give formal rules, discussing the reasoning behind the rules, define the derivative, show a few good examples, and allow students to practice during class, within groups. Send the students from class with some problems to practice finding the derivative, as well as problems asking them to find the slope at a point, in a given context. Ask them to also find the equation of the tangent line at the point. This will require some extension of the problem solving activity since students were never shown how to find the equation of the tangent line, only the slope. Yet, it is within their knowledge base to do so.

**Graphing Linear Functions Using the Slope and the y-Intercept**

For the first stage, discuss prior knowledge. Students have learned to graph by plotting points and by using a graphing utility. We want students to graph using the slope and the y-intercept. Display the graph of a linear function and find the slope of the graph and the y-intercept as a review or reminder of previously learned material.

Give the class the problem. A salesman earns $200 per week plus a 10% commission on what he sells that week. The function for his weekly income is \( I(x) = 200 + 0.10x \). Graph the function using the slope and the y-intercept. Use the graph to estimate the salesman’s income if he sells $5000 worth of products in a given week.

Group students and indicate that they are to discuss the problem in their groups, graphing by any method. After sufficient work has been done on the problem, call the students together...
and discuss the methods they used to graph. Students should have graphed either by picking points and plotting them, or by using a graphing calculator, since these methods are in their knowledge base and in their existing schema. Discuss that we still haven’t graphed by the prescribed method. We return to exploration and designing a new plan, thus practicing control.

For the second stage, guided orientation, distribute the prepared group activity. This activity should ask groups to graph linear families of functions on their graphing calculators. One family of functions should hold the y-intercept constant and vary the slope. The other family of functions should hold the slope constant and vary the y-intercept. After completion of the activity, students should start to notice connections between the equations and the slopes and y-intercepts of the graph.

During the third stage, explicitation, groups should verbalize the relations found. Once groups have done this, bring the class together and discuss the findings of the different groups. Come up with a statement that all groups can agree upon describing these relations.

Send students back to their groups and enter the fourth stage, free orientation. Groups should test the class description found above against graphs of other linear functions. At this time, groups should revisit the initial problem, designing a new plan, and graphing the function using the class description of the relations between a linear function and its graph. They should check this either by picking points and checking that they are on the graph, or by graphing on the graphing calculator and checking both the y-intercept and the slope. The group’s plan should include analyzing the graph to find the week’s salary for the salesman under the given conditions.

Finally, formalize the class findings during the fifth stage. Derive $y = mx + b$ from the slope equation, using the points $P(x,y)$ and $Q(0,b)$. Discuss with the class, using good
questioning techniques, a list of steps by which a linear function can be graphed using only the slope and the y-intercept. Demonstrate the use of these steps. Also at this time, discuss the final answers to the problem, comparing answers and strategies between groups. Allow for students to practice graphing using this method within their groups to ensure of their ability them of their understanding of the process.

Assign skill problems allowing individual practice in graphing by using the slope and y-intercept. Also, give equations in a real-world context, asking students to graph using the newly learned method. Ask them to extend their learning by also finding $x$ given $y$, and $y$ given $x$, in the given context.

V. **How Can my Theory be Used?**

My theory can certainly be applied immediately to classroom instruction. I believe most topics can be taught using this method. As problem solving is important to student learning today, this method embodies an effort to teach problem solving. This method also lends itself to active learning in the classroom, keeping students mentally active in learning, while allowing for a balance of abstraction to add depth to their learning.

I am uncertain of the role this method would play in assessment and research. Certainly, classroom assessment is possible and recommended. The assessment results should be analyzed and teaching and learning strategies altered if the data indicates this. I would be interested in discovering whether this weaving of problem solving, active learning, and abstraction together would increase student learning, given a larger data base on which to collect evidence.