What it Means to Understand Mathematics

If one looks up the word ‘understand’ in a dictionary, one would find a variety of definitions for the word. Consequently, it first seems necessary to choose one definition and then to proceed from there. The definition that I think is most useful for use when discussing the understanding of mathematics is as follows (World Book Multimedia Encyclopedia, Mac OS X Edition, Version 6.0.2, 2001):

To know well, especially: a. to be able to explain, discuss, use, or experiment with. b. to know how to deal with.

This definition, however, leads to an important question: Which of the above is desired for the use of the mathematical knowledge to be understood? Does one wish to be able to explain mathematics to others, to have mathematical discussions with others, to use mathematics to solve problems, or to be able to deal with mathematics in some other fashion? The answer to these questions are important before one can decide what it means to understand mathematics.

I have seen many examples in which one person will say that another person understands a mathematical concept, but in many of these cases the understanding of the persons in the second case appear to be different. For example, I have often seen a gifted student help other students with difficult computations on a homework problem, leading the students being helped to the conclusion that the first student understands the material. On the other hand, most students would say that their instructor in a mathematics class understands the material because the instructor gives a clear explanation of the concepts to be considered. In both cases, one would say that the individual understands mathematics in some form, but one would find it difficult to say that each person’s understanding was of the same nature. Consequently, it seems that there are several different ways in which a person can understand mathematics.

Moreover, it is my opinion that some of these means of understanding require some of the other means of understanding. For example, one cannot convincingly
discuss mathematics without first being able to use or deal with mathematics. Also, an instructor would have difficulty explaining mathematics without first having the understanding required to first have a discussion involving mathematics.

It is for these reasons that I find myself drawn to a theory which involves a hierarchical structure of understanding mathematics. Primarily, I find myself drawn to a theory of levels, most similar to that of Pierre Van Hiele and also somewhat in favor of a set of levels similar to Bloom’s taxonomy. At this point I’m not sure that I could develop an adequate set of levels to describe mathematical understanding, but I do believe that such a set has its usefulness and even find that the levels of Van Hiele and Bloom are useful tools for determining mathematical understanding.

At this point, I should note that I am not suggesting that the phenomenon of understanding mathematics occurs in a discrete fashion. In fact, I believe that the process of developing such understanding occurs in a continuous fashion that is constantly building upon previous understanding. I am inclined to favor the use of levels as a means of representing various points along the continuum of developing understanding. By choosing points along this continuum, it is my belief, that it becomes easier to discuss and gage an individual’s understanding of a given mathematical concept.

**How this Understanding Develops**

To some extent, I explained my views of this topic in the previous section. I believe that to develop understanding at any given level one must first have achieved understanding at the lower levels. Again, I believe that this view is in agreement with the theory given by Van Hiele in his book “Structure and Insight: A Theory of Mathematics Education,” 1986, Academic Press, Inc. In this book Van Hiele stated (pg. 51):
The ways of thinking of the base level, the second level, and the third level have a hierarchic arrangement. Thinking at the second level is not possible without that of the base level; thinking at the third level is not possible without thinking at the second level.”

Because of this view, I find my own beliefs on how mathematical understanding develops to be loosely in agreement with many of the ideas from the theory of constructivism. It seems that in order for a person to develop understanding of mathematics, they must rely upon the prior knowledge and understanding that they have already attained.

Consequently, when developing mathematical understanding for a given concept it becomes necessary to have a starting point. Van Hiele’s starting point is what he calls the visual level and the starting point for Bloom’s taxonomy is knowledge. I don’t necessarily find these to be completely incompatible and would suggest that the starting point be that of recognition. That is, does a person have the understanding to recognize mathematical objects? For Van Hiele this would primarily be the recognition of geometric shapes (perhaps the reason for the use of the world ‘visual’?), but this could also refer to recognizing the presence of three apples or recognizing that a given function is a polynomial based upon the knowledge of the definition of polynomial.

Once one has a starting point, then the means for thinking about and discussing mathematical concepts becomes possible. This then allows individuals to begin to explore the concepts and/or the relationships between multiple concepts. After which point these individuals will become ready to understand a concept at a higher level. Of course, this higher level then allows individuals to think about concepts and/or the relationships between multiple concepts in a different way, which then leads to yet another level.
Example of How this Theory Works

Because I do not yet have my own hierarchy of how mathematics is understood, I will give two examples of how such a hierarchy works by using both Van Hiele’s levels and Bloom’s taxonomy separately. In this way, it is my belief that it can be demonstrated that any adequate hierarchy can be shown to be useful. In fact, many such hierarchies exist and have been used for quite some time.

For these examples I will use the idea of understanding functions. The reason for this is that upon reflection, it becomes evident that the statement that someone understands functions can mean many different things. Does the person simply have the definition of function memorized? Can the person recognize functions and if so, is the recognition for algebraic relations or does the person recognize abstract functions as well? Does the person recognize the properties of functions and the relationships between multiple functions?

Example 1, Van Hiele Levels:

Level 1, Visual: To obtain this level a person first needs to see many examples of functions and then based upon this experience be able to recognize other functions within a variety of relations.

Level 2, Descriptive: The person uses the observations from the visual level to describe the features of a function.

Level 3, Theoretical: The person develops a well-defined definition for the concept of function.

Although Van Hiele gives five levels in his book, I will stop here. Although I find this description to be somewhat inadequate, I do believe it gives good insight into one way in which a person can begin to understand the concept of function. I also think that this example provides an adequate method for introducing functions to individuals.
Example 2, Bloom’s taxonomy:

**Knowledge**: A prior understanding of relationships between objects is called upon.

**Comprehension**: After seeing various examples of functions the concept begins to materialize internally.

**Application**: The person uses the prior comprehension to recognize functions within a group of relations.

**Analysis**: The person begins to study functions and develop ideas about the properties of functions and the relationships between functions.

Again I find this description to be somewhat lacking. For example, where does the person develop the actual definition for function or is this to be given within the knowledge level? Despite this, however, I believe that this example provides a very similar method to the one above for introducing functions to individuals.

How this Theory Can be Used

In order to apply levels of thinking to the practice of teaching, it first seems necessary to determine which level is desired for the student to attain, for each concept to be taught. For example, in an algebra class it would be adequate for the students to only reach a lower level for many of the concepts taught and it would definitely be adequate for the students to reach a level below one requiring theoretical thought throughout the curriculum. Concepts such as the properties of logarithms and the exact behavior of the graph of a polynomial may only require understanding at a descriptive level, as a higher level would require higher levels of mathematics. In comparison, for students in a calculus class, a higher level leading towards theoretical understanding
should always be the goal. In a calculus class the goal is always for students to think beyond a descriptive level and in more theoretical terms. Since, however, many calculus students are potentially pursuing a degree in mathematics or science, it is desirable for them to begin to understand the concepts under the rules of formal logic. Thus once the appropriate level has been determined as a goal for students to reach, the teacher should develop appropriate means for attaining that goal and whenever possible, look for opportunities to challenge the students to think at the next level.

The next obstacle for a teacher to overcome is that of determining at what level students already understand prerequisite material. If the students understanding is inadequate, then the teacher needs to determine if the student is capable of continuing within the class or if the student needs to return to a lower class and master the material at the desired level. Admittedly, this is difficult in many ways. First, this is difficult information to attain from every student within the course of the classroom activities. It is necessary, however, that an attempt is made to understand what students do or do not bring to the learning situation. Second, it is often difficult to convince students of the need to take a lower course. In many cases this is an impossibility and the teacher must move on with the knowledge that the student is unprepared. The teacher may attempt to raise these students' understanding, but should only do so without hindering the other students abilities to reach the desired level.

Once these obstacles have been overcome, the teacher should be ready to begin to develop lesson plans that incorporate the theories of the levels of thinking. Conveniently, the use of levels gives a teacher a good place to begin. The teacher simply needs to begin at the first level, which I have called recognition. To obtain this the teacher must allow the students to observe enough examples so as to begin to recognize certain objects and concepts. For example, algebra students can be shown many examples of equations so that they may visually be able to identify what an equation is. In this situation obtaining the first level of understanding may only require a
short presentation. Once this level is established, students are ready to move on to the next level of understanding.

Well planned exercises and classroom discussions are then necessary to guide students towards higher levels of understanding. These levels may be ones that allow an individual to discuss mathematics. For example, in an algebra class students can be given equations and then be asked to evaluate the equations for given values of x (assuming x is the variable in the equations). Some of the values may be solutions to the equations, while others are not. In this way students become aware that solutions to equations exist and may begin to question how these solutions can be found. What is then required for the students to move on to the next level is a class discussion in which the students may exchange ideas about what they have observed and may present possible properties of the concept. It is important at this stage that the teacher guide the discussion so that students stay within the bounds of the topic and so that the students do actually approach the next level and begin to understand the desired properties of the concept to be taught.

Next, the teacher may wish to bring the students to levels that allow them to use mathematical concepts. For example, in an algebra class, students can be asked at this stage to attempt to solve equations on their own. This stage is the point in which carefully chosen homework problems or classroom exercises become useful. It is important to choose problems that are based directly upon the discussions of the previous level, allowing students to develop a better understanding of the properties in general settings.

Finally, if more advanced levels are desired, this can be achieved through classroom exercises and appropriate homework problems. These problems should challenge students to use the concepts they have developed in somewhat unfamiliar situations. In this way the students can integrate their new knowledge into many situations and be able to apply the skills to many different types of situations. Once this
has been accomplished the students will then be able to memorize the concepts and techniques and will have them ready when they become necessary in future situations.

From the above discussion we can see that in order to begin to understand something at a higher level, it is first important to understand it at a lower level. The reason being that the tools used to move on to the next level are based in the previous level. Thus through some method or another, to be determined by the teacher, it is necessary to test whether or not the students understand the concepts at a lower level before moving on to the next level. This can be done in a number of ways, some suggestions are through careful class discussions in which every student needs to participate or through carefully written quizzes that only test at the lower level. Of course, there are instances when some students must be left behind due to the amount of time available. In these instances, however, the students who are lacking understanding should be encouraged to seek extra help before the next concept is presented.

This statement then brings up the next important point related to this method of teaching. In most classes there is a time constraint and there are demands on what material should be taught to students within a particular class. Also, it is important to realize that the methods described above take much more time than conventional styles of teaching. For this reason it becomes necessary to determine exactly which concepts are needed to prepare students to move on to the next level and to satisfy any requirements on curriculum. Once this has been done, any material that is not necessary should be left out. It needs to be understood that the time required for students to learn mathematical concepts at a high level is greater than the time given in conventional classes. In most mathematics classes taught at the university level today, the material is simply presented during lecture, coupled with a few examples; and leave little time for students to develop their own understanding of the concepts. Students are then given homework problems similar to the problems
presented in lecture and finally are tested on problems similar to their homework. This method of teaching does not raise a students level of understanding, nor does it test at which level a student understands the concepts. It simply teaches students to memorize specific procedures and techniques and to reproduce them on homework and exams, in order to receive a passing grade in a class. This should not be the goal of any academic institution, nor the goal of an individual educator. There need to be some goals set by the teacher as to what concepts the students should understand and at what levels the students can be taught to think.

For it is not just desirable for students to think about mathematics at higher levels, but to be able to think about actual life situations at these levels. In this manner, even if a student does not plan to ever take a mathematics course again, the student will have achieved the ability to think at a high level within whatever situation they find themselves. It has been said that mathematics courses do not exist for the sole purpose of teaching mathematics, but to teach people how to think. Van Hiele mirrors this view when he describes his levels of thinking with the statement, “the above classification is suitable to a structure of mathematics and perhaps mathematicians will be able to work with it. However, our aim was the improvement of thinking” (pg. 53). Consequently, students should be given the time required to think about the concepts themselves, so that they may be able to develop the skills necessary for thinking at higher levels.