Assessing conceptual understanding of rational numbers and constructing a model of the interrelated skills and concepts

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In my synthesis paper, I described the struggle that students have understanding rational numbers. In that paper, I proposed that we need a system for identifying students’ strengths and weaknesses dealing with rational numbers in order to jump the hurdles that impede instruction. **We need a model for describing learning behavior related to rational numbers – prerequisite skills and development of rational number sense – that is dynamic and allows for continuous growth and change.** It would inform us of the important background knowledge that students bring with them and the prior experiences that influence their level of understanding. It would further enable us to assess students’ current levels of understanding in order to prescribe the necessary instruction to continue forward progress. Designing a method for assessing students’ conceptual understanding of rational numbers that has this potential is a challenge. In this paper, I will discuss where the call for conceptual understanding stems from in the recent past, what has already been done involving assessment of conceptual understanding, what reach has revealed about acquiring skill and number sense with rational numbers, and describe a plan using this information for developing a continuum of rational number skills and concepts.

**Background on reform in mathematics as it relates to conceptual understanding:**

National assessments and reports often act as a jumpstart for research agendas, curriculum development, and professional development training. Analysis and assessment of student learning weaves its way into all three categories as the message of current reform in mathematics becomes clear. Assessment is not something done to the students but for the students and therefore must inform instruction. It is important to recognize that the research I’ve proposed has not come out of nowhere and is not a new idea. What is being suggested is to bring it all together in a practical way. Briefly highlighting various assessments and reports that have identified and highlighted the importance of conceptual understanding enables one to trace back to the “hatching” of the idea. In addition, outlining the course that got us to where we are today, trying to
determine what it means to understand something and how understanding can be assessed assists us in continuing that course of action in the right direction.

In 1980, recommendations were made by the National Council of Teachers of Mathematics for reforming mathematics instruction in An Agenda for Action. These recommendations were based on results of the second National Assessment of Educational Progress (NAEP) and on data collected by the National Science Foundation (NSF) largely from a study called “Priorities in School Mathematics” (PRISM). Specifically in the area of fractions, NAEP contended that students’ inability to compute with fractions was the result of dependence on rote memorization of algorithms and a focus strictly on routine problems. Among eight recommendations, An Agenda for Action called for problem solving to be the focus of school mathematics in the 80’s and basic skills in mathematics to be more than computational fluency. The fourth NAEP showed improvement, but indicated that mathematics instruction still lacked depth, particularly in the area of conceptual development of fraction and decimal skills.

In 1983 the National Commission on Excellence in Education released a report, A Nation at Risk, which identified, analyzed and made recommendations for addressing problems and deficiencies in American education. The recommendations where made in an attempt to battle diluted content, low expectations, less time spent in school than other nations, and a serious shortage of qualified teachers. The recommendations involved raising high school graduation requirements in mathematics, creating measurable standards based on higher expectations, adopting more rigorous curriculum that is not a mile wide and an inch deep, and improving teacher preparation. The idea of delving deeper into content indicates a shift from strictly procedural skill toward an inclusion of conceptual understanding. With this shift comes a need for new standards as well as a way of assessing this type of understanding.

While A Nation at Risk focused on secondary education, Everybody Counts, a report released in 1989, studied mathematics education as one system from kindergarten to graduate school. The report was built on the philosophy that children are naturally curious and come to school with meaningful mathematics already in existence in primitive stages. The school experience causes them to perceive mathematics as rigid and procedural because of excessive emphasis on computation and rules. The study found
that “teachers tend to teach only what is in the textbook and students learn only what is in the text” (p. 45) rather than experience real mathematics that reinforces their curiosity. The report states that in elementary school the focus should be on number sense that develops from concrete experiences. In secondary education, the focus should be on the transition from concrete to conceptual mathematics, from numbers to variables, and increase the development of number sense. The key word in the above statement is transition. The movement from concrete to abstract experiences between elementary and secondary school is often more like trying to leap the gorge rather than cross a bridge to get across. Some students never make it. One of several changes identified in the report is a shift toward not allowing students to sit and passively absorb rules and procedures. Rather, an environment that encourages students to explore, understand, and communicate mathematics needs to be created.

More recently, a report to the nation from the National Commission on Mathematics and Science Teaching for the 21st Century titled Before It’s Too Late was released in 2000. The members of the commission where charged with the task of investigating and reporting on the quality of mathematics and science teaching in the nation and considering ways of improving recruitment, preparation, retention, and professional growth for teachers. The primary message is that student achievement in mathematics and science must improve. It is vital to our nation’s growth and defense. Second, the most direct route to this change is by improving teaching. The core of math and science teaching should be inquiry. This requires hands-on approaches and real-life situations. It teaches students not only what to learn but how to learn. Identifying how people learn and sharing this with teachers becomes a central part of improving teaching and consequently improving students’ achievement.

At about the same time Before It’s Too Late was released, the National Research Council released a report that presented findings on how education should be conducted and emphasized that teachers are key to change. This report titled How People Learn: Bridging Research and Practice highlighted that students enter with preconceptions and recognizing this, the teacher can use them to shape understanding that reflects the concepts and knowledge. The report also states that students need a deep foundation of factual knowledge and a solid conceptual framework. In addition, students need to
develop strategies to monitor their own understanding and progress. The goal for teachers and students then becomes to better understand what this understanding looks like.

Although national assessments and reports often act as a jumpstart for research agendas, curriculum development, and professional development training, that doesn’t necessarily ensure that education professionals in the field put the information into action. Change is difficult in the American educational system. Unlike other countries that have a strong “top-down” approach to education, curriculum development in the U.S. involves teams of local teachers meeting in the summer to write new mathematics curricula. “These teacher teams usually have little training in the complicated process of curricular development, little or no help in coping with changing needs, and little to fall back on except existing textbooks, familiar programs, and traditions” (Everybody Counts, 1989, p. 77). Although teachers should be a part of the decision making regarding curriculum, this is nearly an impossible task with nearly an impossible time frame. Teachers can’t possibly become the experts in every content area in order to make curriculum decisions based on research. Thus putting the right tools into the teachers’ hands becomes essential.

Following the history of national reports provides an interesting and important foundation of knowledge regarding what defines quality instruction, what assessment of this instruction should look like, how assessment further drives instruction, and how the sum of this breaks with “tradition”. This is why there is a need to “package” programs that reconfigure the familiar, tie in new information about student learning, and provide teachers with the big picture of how it all fits together. To create an environment that is not exclusively procedures and rules and to encourage students to explore, understand, and communicate mathematics, teachers need better tools for analyzing and assessing that type understanding. In addition this information will better prepare teachers to assist students as they transition from concrete to abstract concepts and maps out what the development “understanding” looks like.

**Background on conceptual vs. procedural understanding**

The resulting essential questions thus become: What constitutes procedural skills? What defines conceptual understanding? How can they be assessed? According to Hiebert (1984), students acquire procedural (form) and conceptual (understanding)
knowledge independently. Students that do not make the connection between the rules/procedures and an understanding of the concept that drives the rules and procedures may suffer serious consequences in their learning of mathematics. He states that by the third or fourth grade most students’ greatest concern is about form. This can have serious consequences. For example, a lack of conceptual understanding in regards to partitioning can be detrimental if overlooked by students attempting to develop decimal concepts. It is considered an essential concept when learning about fractions and becomes more complicated when involving decimals.

Wearne and Hiebert (1988) conducted three studies focusing on procedural versus conceptual knowledge. The studied the skills students possessed working with decimal symbols and the meaning tied to the concept. The first study revealed that students had a lack of connection between concepts and procedures and as a result could only solve routine problems. This finding confirms what the second NAEP results indicated was occurring in the early 1980’s. Wearne and Hiebert’s study demonstrated the consequences of memorizing rules for the use of decimal symbols and supported the idea that “without conceptual meaning to support the rules, they are frequently forgotten or distorted and rigidly applied.”

There are competing theories concerning the development of conceptual and procedural knowledge. Much of the past research and theory focuses on which type of knowledge is first to emerge (Rittle-Johnson & Aliabali, 1990). This has been debated for some time. The newest theory to emerge proposes that the development is simultaneous in a hand-over-hand fashion and that one influences the other (Rittle-Johnson, Siegler, & Alibali, 2001). Central to the numerous theories is that students that construct their own knowledge of a concept will be more able to use that knowledge than a student that simply memorizes rules or follows a series of steps (Behr et. al, 1984).

Regardless of how it is developed, research shows a link between three kinds of mathematical knowledge and individual differences in fraction computation, word problem, and estimation skills (Hecht, 1998). The three kinds of knowledge are procedural knowledge relating to fractions, conceptual knowledge relating to fractions, and math fact knowledge. Procedural and conceptual knowledge where found to be
responsible for variance in fractions computation and fraction word problems. Only conceptual knowledge was found responsible for variance in fraction estimation.

Furthermore, current research supports what earlier reports found, that changes in the instruction of teachers resulted in changes in student achievement (Fennama et al., 1996). A longitudinal study of teachers that participated in a Cognitively Guided Instruction teacher development program studied how teachers evolved from a focus on procedures to an emphasis on concepts and problems solving. It involved using children’s thinking by engaging them in problem solving activities. In the case of every teacher, there was an increase in concepts and problems solving achievement. Although the focus of the teachers was on concepts and problem solving no change in computational performance was reported. The researchers found the “study provides strong evidence that knowledge of children’s thinking is a powerful tool that enables teachers to transform this knowledge and use it to change instruction” (Fennema et al., 1996, p. 432). Evidence that it we need instruments to help teachers understand the thinking of their students as they learn mathematics.

**Background on understanding of rational numbers**

A huge amount of research and publications have come from the Rational Number Project. Funded almost continuously since 1979 until 2002, it is the longest lasting federally funded multi-university project in the history of mathematics education. To encourage continued research in this area, all the publication related to the project have been accumulated and posted on their website (The Rational Number Project). Here one can find over 90 publications spanning 23 years! (Unfortunately I was not aware of this website until recently so I’ve only been able to read a fraction of the articles available at the time of this paper!)

Research indicates that before receiving formal instruction on rational numbers, students possess practical knowledge that they are able to use to solve real-world problems (Kieren, 1988; Mack, 1990). **Informal knowledge**, knowledge that students bring to school and to instruction, and **previous knowledge**, knowledge that may be informal or formal but learned prior to the current learning experience, can both help and hinder instruction. Research shows that students have misconceptions that stem from their previous knowledge that interferes with their understanding of rational numbers.
Some of these misconceptions stem from the improper application of their prior knowledge, meaning they apply it in a way that they shouldn’t. A number of studies have been done to offer explanation for the difficulties students have.

Applying acquired notions of whole numbers to developing notions of decimals is a common occurrence among middle school students. The Principles and Standards for School Mathematics states, “Students must learn mathematics with understanding, actively building new knowledge from experience and prior knowledge” (National Council of Teacher of Mathematics, 2000, p.11). Students’ prior knowledge of whole numbers, however, can interfere with the development of decimal fractions (Sowder, 1997; Resnick, Nesher, Leonard, Magone, Omanson & Peled, 1989; Sackur-Grisvard & Leonard, 1985). The fact that whole number conceptions can both support and interfere with decimal fraction development may be due to the inherent relationship between whole number and decimal knowledge, which are in some ways similar and some ways different (Sackur-Grisvard & Leonard, 1985; Resnick et. al., 1989). For example, knowledge of decimal fractions is similar to whole numbers in that the value of the digits increases as they move right, but different in that as the decimal portion moves farther away from the decimal point its value decreases and for the whole number, it increases.

Students made several common, temporary errors by falsely identifying “rules” to follow when comparing decimal numbers. Sackur-Grisvard & Leonard (1985) identified several rules. The first false “rule” that students apply is to choose as smaller the number whose decimal portion is smaller (e.g. 1.6 is smaller than 1.16). Rule two usually appears after rule one and it states that students choose as smaller the number that has more digits in its decimal portion (e.g. 10.43 is smaller than 10.2). Students in this case recognize that as they get further from the decimal point the places decrease in value. Sowder (1997) observed that students were confused about the meaning of the symbols and showed a lack of understanding of place value involved in manipulating decimal numbers. The students in her study supported research done previously that showed students lack understanding of decimal numbers. A solid understanding of place value appears to be an essential building block for decimal numbers (Sowder, 1997; Heibert, 1992).

Whole number notions interfere with how children think about fractions as well (Behr et al., 1984; Post et al., 1985; Post & Cramer, 1987). Applying knowledge of whole
numbers, students comparing unit fractions such as ¼ and ½ would choose ¼ as the greater fraction because four is more than two. Students also mistakenly reason that 4/7 is equal to 2/5 because the whole number difference between the numerator and denominator is the same.

The same use of incorrect rules and procedures plagues work with percent as well, indicating an emphasis on rules and procedures rather than the concept. In interviews, students appeared to lack confidence in the ability to talk about the multiple and equivalent forms of rational numbers and the relationship among fractions, decimals, and percents (Gay & Aichele, 1997).

Research indicates that the first and main instructional approach for teaching fractions which receives almost all the attention in textbooks is the part to whole model and the teaching mainly focuses on procedural skills. (Behr, Post, and Silver, 1983; Behr, Post, and Lesh, 1984; Sowder, 1992). While the learner must understand this model, exposure to other interpretations of rational numbers is critical. Other interpretations of fractions would include: fractions as measures, quotients, operators, and decimals. Concepts of partitioning, equivalence, ordering and performing operations are important to developing rational numbers skills and concepts as well.

In addition, the part-whole approach can be modeled as a region or area (taking a whole rectangle and partitioning it into equal parts), but students also need to experience partitioning activities that use set models and focus on objects that are both divisible and not divisible (for example 7 candy bars among 5 people). The set model involves discrete or countable measures. The number line model on the other hand differs and is more complex in nature because it is a continuous measure. Research indicates that the number line model provides a greater challenge for students (Riegel, 1991).

The most basic understanding of part-whole relationships, the meaning of the numerator and denominator, is often overlooked when teaching fractions (Sowder, 1995). The denominator indicates the size of the parts and the numerator indicates the number of parts being considered. The result is why children use existing whole number knowledge when they shouldn’t and make the mistakes described previously in the section about whole numbers.
Combining the features of place value and partitioning are necessary for creating meaningful notions of decimal fractions. Hiebert states (1984) “decimal representation of number is subject to the added constraint that a whole must be partitioned into exactly 10 equal-sized pieces.” A lack of understanding of these concepts results in students writing the fraction \(\frac{1}{4}\) as 1.4 as a decimal.

Another property that is important to developing and understanding of fractions that is often neglected is that a fraction represents a single quantity (Mack, 1990; Sowder, 1995). Students often look at the symbolic representation of fractions as two independent whole numbers rather than as a specific quantity. When thinking of the fraction \(\frac{3}{8}\) using a part-whole area model, the 3 and 8 together refer to a specific part of a whole rather than the 3 representing one quantity and the 8 representing another, unrelated quantity. Students operating in this mode will frequently add the numerators and the denominators of fractions together when performing addition of fractions.

Another important property to consider is the multiplicative relationships that exist when a fraction is interpreted as an operator. It often presents students with unique challenges. Post (1989) suggested that teaching students about multiplicative relationships involving rational numbers would help them overcome inappropriate application of additive patterns in exchange for multiplication patterns when dealing with equivalent relationships. Proportional reasoning skills are related to this concept and are found to be influential in the development of understanding of rational number.

Number sense with rational numbers is often used as an indicator for conceptual understanding. A traditional textbook emphasizes equivalence of fractions for the purpose of comparing and ordering them, which involves a set procedure. The use of benchmarks on the other hand is often an indication of fractional number sense when comparing and ordering fractions (Markovits & Sowder, 1994). The use of benchmarks could also be applied to decimal and percent.

Researchers indicate that the use of manipulative materials can help students distinguish between whole-number and rational-number concepts (Behr et al, 1983; Post &Cramer, 1987). Modeling with manipulatives has been shown to create confusion yet at the same time facilitate learning. This occurs when students the model comes into
conflict with the student conception, eventually resulting in a clarification of the erroneous notion and creation of a new more powerful conception.

When assessing a student’s understanding of rational numbers many properties or features of fractions, decimals, and percents have been identified as significant and need to be considered. In addition, how well these ideas are connected or related to one another is important. Finally, the type of understanding, the amount of procedural versus conceptual understanding regarding rational numbers is important too.

**Existing models:**

Many models exist and many researchers have attempted to address what it means to “understand” a concept. Theories and models range from simple to quite sophisticated. Each offers important considerations when attempting to design a method for assessing students’ conceptual understanding of rational numbers that can be administered by teachers, is practical in nature, and informs instruction. Aside from SOLO taxonomy which I referred to in my previous paper and the models we’ve studied in class, there are several new ones I’ve been exposed to during my reading.

Lesh, Post, and Behr (1987) approached “understanding” from three perspectives. The first perspective considers the learner’s ability to identify a mathematical concept in various representational systems. Second, the learner has the ability to model the concept in one of the representational systems. Third, the learner has the flexibility to move from one representational system to another. The representation systems they refer to are: real-life situations, manipulative models, diagrams or pictures, words (verbal expressions), symbolic notation.

A theory by Wearne and Hiebert (1988) suggests a strict sequence of five cognitive processes that combine to create competence with decimal symbols. These five processes are: connecting, developing, elaborating, routinizing, and building. They found that instructors could assist students in making meaning of symbols by creating a link between the symbol and a meaningful referent. If this connection was made students could use this understanding to tackle decimals in a new context. Finally, there was evidence to show that semantic analysis of decimal tasks were complicated by and blocked by routinization of rules. Furthermore they found “routinized procedures
insulated these students from assimilating new information or constructing new approaches to solve problems”.

Lampert (1986) proposed a model consisting of four ways of “knowing” and further suggested that instruction must enable students to make the connection between the various stages. The four types of knowing are: Intuitive knowledge, concrete knowledge, procedural or computational knowledge, and finally conceptual knowledge. Intuitive knowledge is tied to the content and represents what the learner has the ability to do initially. Concrete knowledge is the ability to manipulate objects to solve a problem. Procedural or computational knowledge is the process of applying rules and procedures involving symbols and symbol manipulation to solve problems. Conceptual knowledge, also referred to as principled, is the ability to apply knowledge to novel problem situations.

Identifying and defining them separately provides a possible explanation as to how children can use procedures without any notions about why they work. As many reports have stated students are introduced to procedural knowledge prior to any experience with concrete models and are sometimes never introduced to any type of knowledge other than procedural. The obvious result is students that can manipulate symbols but have no idea if their answers make sense and that don’t recognize obvious errors. Assessing students understanding of rational numbers should address various ways of knowing as this research suggests.

Another model involves not only the identifying the types of knowledge that students possess but also considers the growth of their understanding. Pirie and Kieren (1994) committed several years to the development of just such a model which was subsequently revised many times (Pirie & Kieren, 1989; Pirie & Kieren, 1990). The significant piece of their model is that, “It is a theory of growth of mathematical understanding as a whole, dynamic, leveled but non-linear, transcendentally recursive process”. Their model involves eight levels labeled within nested circles, an interesting way to represent the levels. Each circle contains all of the prior circles nested within it and is rooted in all subsequent circles.

Starting with the most inner circle the levels are: primitive knowing, image making, image having, property noticing, formalizing, observing, structuring, and
inventising. I will describe them briefly in order to understand how this model compares to other models. Primitive knowledge is the beginning of growth of understanding and represents what the learner can do initially. Image making involves using prior knowledge in new ways (I’m under the impression this is a more concrete level). Image having is when the learner is freed from the need to use physical objects to solve the problem and instead uses a mental construct. At the level of property noticing, one combines aspects of the developing images to build context specific, relevant properties. The fifth level, formalizing, involves abstracting or generalizing a method, definition or algorithm that would work in any case. Observing, the sixth level, occurs when the learning coordinates the formal activities to construct theorems. The seventh level, structuring, then involves further coordinating the theorems to see them as inter-related and to use them to justify or verify. The final and outermost level is referred to as inventising. This is the level that represents a completely structured understanding which allows the learner to generate new questions with the potential to grow into new concepts.

An important feature of their theory involves three “don’t need boundaries”. These are places in their model where students could work without needing previous forms of understanding. For example, there is a “don’t need boundary” between observing and structuring. Having mathematical structure doesn’t require meaning from any of the inner levels. One could prove theorems without any notion of what they really represent. This is similar to Lampert, although the levels here provide greater detail, in that it again offers a possible explanation as to how children can use procedures without any notions about why they work. This is significant when considering how a student can be at one level without having experience previous levels.

Another feature of this theory that can inform not only assessment of understanding but instruction as well is the activity of “folding back”. This activity results from a problem introduced at one level which cannot immediately be solved. This requires the learner to “fold back” to an inner level to deepen the existing, inadequate knowledge. Returning to the inner level reveals that it is not the same as the original level because it is now informed and shaped by the knowing that is experienced in the outer level. Folding back results in a deeper understanding.
**Precis (principles)**

What does my continuum type of assessment have?

1. It is necessary to identify key concepts and skills that are required for the learning of rational numbers with understanding. They are the items in bold print in the above review of literature. The list is a beginning. It is not exhaustive as I have more reading to do.

2. It is also necessary to construct a model of how these concepts and skills are related to one another. This model would likely be best represented by a continuum, rather than a series of levels or steps (similar to what Pirie and Kieren have constructed) that –
   a. Identifies concepts and skills that would be considered “prerequisite” such as whole number concepts and skills and place value. Susan Empson (1999) has done research on the development of fraction concepts in a first-grade classroom. Her article identifies many of the same concepts and skills already identified and provide tasks to consider.
   b. Identifies concepts and skills that are directly related to rational numbers such as partitioning, multiple representations of equivalent rational numbers, and the meaning of symbolic representation of fractions.
   c. Maps out an order that these skills are acquired – this will likely involve some overlap that reflects a range of time when the skill/concept is developed. This is similar to what Pirie and Kieren refer to as folding back.
   d. Coordinates all of these important ideas and explores how they are connected and related

3. It would require me to assess and interview students that are at a variety of stage of development to identify what they can and cannot do in order to be able to identify key concepts and skills and relate them in some way.

4. It would enable the instructor evaluate what necessary skills/concepts a students has acquired and what still needs to be mastered.
Development

Like most good research requires, the articles I read about this topic focused on one small component of learning rational numbers. It is now necessary to research and describe how these highly specialized concepts are related and together build a strong foundation for understanding rational numbers. Taking into account (or beginning to) all that has been identified as important factors and attempt to put order and reasons to them is the next logical step. It creates a much-needed “big picture” of the understanding of rational number.

It reminds one of the story Seven Blind Mice, by Ed Young. The story is based on a classic Indian tale. Seven blind mice encounter something strange by the pond. One by one they go to investigate and each returns with a different theory. One says, “It’s a pillar”. Another says, “It’s a fan”. The seventh mouse takes his time investigating. He runs on top of the object and back and forth. He finally returns with the right decision. The moral of the story is that “Knowing in part may make a fine tale, but wisdom comes from seeing the whole”. In order to really understand rational numbers we need to put all of the pieces of research together to form a cohesive, coherent, well-articulated package.

As I see it, I now need to accomplish four very large tasks. First, I need to either select a model or come up with my own. Second, I need to come up with assessment questions or tasks that get at each of the concepts and skills identified by research as important to the development of rational number skills and concepts. Third, I need to administer the test or interview to a group of students and map their results to the model. Fourth, I need to look for patterns in the data to see if I come up with any answers about how these skills and concepts develop in relation to each other.

Each of the above concepts and skills in bold print are important to the development of understanding of rational numbers. My goal is to take an existing model, such as Pirie and Kieren or to come up with my own model but to identify how/where these concepts and skills fit specifically into the model. What rational number skills and concepts precede others? Which rational number skills and concepts can be developed without boundaries? Which require folding back? Many of the articles I read provided questions and tasks to assess the various specific concepts and skills. I plan to filter out
good questions to match with specific concepts and skills to create my own assessment tool.

I plan to include questions that assess students understanding of multiple representations: real-life situations, manipulative models, diagrams or pictures, words (verbal expressions), symbolic notation to see how there relate to the various rational number skills and concepts.

Hiebert & Wearne, Lampert, and Pirie & Kieren, constructed different models, but all offer a possible explanation as to how children can use procedures without any notions about why they work. My assessment needs to contain items that assess procedural as well as conceptual understanding. Hiebert points out that the connection between the rules/procedures and an understanding is significant. My assessment needs to be able to measure this as well. This might be done by asking students to explain the concept that drives the rules. This would get at the why or how it works rather than just how to do it.

Many of the articles I read provided tasks, both written and interview that explored various properties and features of rational numbers. I plan to work my way through the articles again, this time looking for tasks that measure the depth of understanding, particularly in the area of conceptual development of fractions as an area of weakness indicated by NAEP. Perhaps finding examples of NAEP questions might assist me with this as well.

Again, many of the articles I read provided tasks that focused on four types of knowing are: Intuitive knowledge, concrete knowledge, procedural or computational knowledge, and finally conceptual knowledge. I plan to also look for items that will measure concrete knowledge as well as abstract knowledge to pin down a way to assess a transition between the two. My hope is that I can study these things separately but also find tasks that overlap in several ways so that I can assess conceptual understanding and at the same time look at concrete and abstract thinking.

Finally, the notion of “folding back” from Pirie and Kieren’s model (1994) informs my research. If levels appropriate for the purpose described in the introduction can be created and the exact characteristics that exist at that level are identified then students identified as struggling at a certain level can be instructed to “fold back” and be
given activities to enhance their understanding at a deeper level in order to overcome their inadequate knowledge.

This is a huge task. At this point, now that it is all on paper in front of me, I don’t even know if it is possible. All that I know is that it is necessary. Way too many students have gaps and inadequate knowledge related to rational numbers that impedes further progress. We really need to know what understanding looks like, what students that are successful in reaching this understanding know that others do not, and how this all fits together. It is a puzzle that will require one piece at a time. Lucky for me I always liked puzzles!
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