INTRODUCTION

Proof. What is it and why does this simple term cause such a stir among mathematics educators and mathematics students? If you were to ask a young child to prove a mathematical fact, they would be happy to show you many examples of how it works. This does not constitute a proof but it is a step in the right direction. If you were to ask a high school student or first year college student to do a proof, you will most likely be met with groans and feelings of disgust. Students at this age have probably encountered proof in a geometry class where they were expected to follow a strict format without much freedom to express proofs on their own. However, if you were to ask a mathematician about proof they would begin to tell you about how beautiful proof in mathematics can be.

Proof has always been a topic of interest for me. In high school geometry and my first year of college, I too did not understand proof. I felt like many other students, frustrated by the fact that we were asked to prove theorems that the book had already told us were true. It was as though the instructor was playing magical games on the chalkboard and all of the sudden we had a proof. However, as time progressed, I began to see the beauty of proof. Then, mathematical induction introduced me to the power of proof. In this paper I hope to address the concept of proof, how it relates to understanding and the implications for mathematics education.
BACKGROUND

In the 1950’s and 60’s proof played a significant role in mathematics education. Then in 1989, the National Council of Teachers of Mathematics (NCTM) deemphasized proof and replaced it with reasoning. Following this, mathematics educators began to see that students had difficulty with proof because they had little contact with it. In response, NCTM in the 2000 standards, elevated proof to a standard, emphasizing that it should be part of all students mathematical experiences (Knuth). Schoenfeld states proof is inseparable from mathematics. It is essential in communicating, doing, and recording mathematics (153).

Throughout most of the history of mathematics education, proof has been more of a topic of study instead of a way to understand mathematics (Knuth 73). In addition, proof has only been limited to the college bound student or the student enrolled in geometry. It has been thought for much time that the average student had no real need for proof (Knuth 61). However there are many faces to proof which can be implemented at different levels to help increase students understanding of mathematics. Various authors have suggested that proof has value for students in the following areas:

- To verify a statement is true
- To explain why a statement is true
- To communicate mathematical knowledge
- To discover or create new mathematics
- To systemize statements into an axiomatic system (Knuth 63).
Typically most students are only introduced to verifying that a statement is true. In geometry class, proof tends to follow the pattern of definition, theorem, proof (Almeida 869). When following this method the students see no need for the remaining uses of proof. There is no room for creativity for the student who is forced to use this method. In addition the students already know that the theorem is true and many do not see the point of proving statements just for practice. Artificially reconstructed proof provides little or no insight to students about the process or motivation for proof (Sriraman 2). In addition proof is considered a valid method to convince mathematicians of truth but it does little to convince kids (Battista 48).

The reform movement in mathematics has elevated the value of proof in mathematics education. In the 1989 standards, proof was all but eliminated from the standards but then in 2000 proof was elevated to a standard. Much of the reform curricula encourages students to explain why statements are true, to communicate mathematics, and to discover or create new mathematics. Typically students are posed with a problem and then encouraged to develop the mathematics out of the problem. The reform movement is calling for changes in all mathematical curricula as well as the instructional practices of teachers. In turn students will have a much stronger appreciation of proof.

Proof should play a more central role for all students Pre K-12 (PSSM). Even informal proof for lower level students would be a move in the right direction (Knuth 73). The new reform curricula is created by posing problems, analyzing examples, making conjectures, offering counterexamples, revising, and eventually arriving at the theorem (Battista 48). This pattern followed in the new mathematics curricula is more in line with how mathematicians view proof: intuition, trial, error, speculation, conjecture, and proof.
This allows students to have more control over the material and more ownership of the concepts. By using their own creativity and problem-solving strategies to reach a conclusion, students will better understand the mathematics and the power of proof.

The reform curriculum also refutes the idea that proof is only for the smart kids. Informal proofs that are based on empirical evidence are good for all students (Knuth). The process of posing a problem and allowing the student to “play” with the mathematics promotes proof as an explanatory tool. It allows proof to promote understanding, which is one of the aforementioned values of proof. The reform movement also supports a great deal of flexibility in how a proof can be written. Typically in a high school geometry course, students are asked to write proofs in a particular format. Each proof should be written in two-column format with the premise in column 1 and the definition or theorem in column 2. Each statement flows after the other in a crisp and deductive format until the desired outcome is reached (Sriraman 2). However, Schoenfeld argues that there is a great deal of flexibility in how a proof can be written. The goal is to produce a chain of argumentation in which coherence and correctness are what matters (Schoenfeld 153).

Too many times we forget that students need to discover the mathematical truth. Then by allowing them to prove this truth, it will support understanding of mathematics.

Studies have found that formal deduction among students who have studied geometry in high school is nearly absent (Battista 49). Since this is the case, we need to look deeper into how one develops the concept of proof.

**PRECIS**

According to Piaget, in order for students to gain the ability to construct a proof, one must progress through certain stages of learning. At stage 1, a child is non-reflective,
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unsystematic, and illogical. Students at this level will explore randomly without a plan. They will be unable to generalize from one example to the next.

Stage 2 students will begin to try and justify their predictions. Gradually students at this level will begin to establish relationships, anticipate results and think logically about premises that they believe in. Students at this level are still not ready to conquer the concept of proof but are beginning, on their own, to use some informal reasoning to justify conclusions.

At Stage 3 students begin to see that because a statement is always true implies that it necessarily must be true. Students in this stage are capable of formal deductive reasoning and can operate within a mathematical system. In addition, students at this level will be able to recognize when something does not belong to a certain set because it does not meet the prerequisite properties.

While Piaget’s theory deals with understanding in general, the van Hiele levels are specifically applied to geometry. We can use his ideas to further explore how our students can develop proof specifically in geometry.

Students who are at the first two van Hiele levels are not ready for proof. Level 1(Visual) students typically reason about geometric shapes based on their appearance. Their reasoning is based on previous examples they have been shown. Students at this level will not be able to characterize shapes that are shown to them in an unusual format. Students do not doubt the validity of their observations so they see no need for proof. At Level 2(Descriptive/Analytic), students begin to reason experimentally. They begin to notice properties and categorize shapes based on these properties. Typically these
properties are noticed by looking at a group of similar shapes not by looking at non-examples.

At Level 3 (Abstract/Relational) students begin to reason logically. They can formulate definitions, distinguish between necessary and sufficient conditions, and even begin to present logical conclusions. Deductive reasoning can begin at level 3 when students begin to make logical relations between properties. As students progress for Level 3 to 4 (Formal deduction) they begin to reason formally and logically. They are capable of constructing original proofs by producing a sequence of statements to logically justify a conclusion. Finally at Level 5 (Rigor/Metamathematical) students will reason formally about mathematical systems. Most high school students will never reach level four. (Battista 50)

As we begin to think seriously about how to integrate proof more effectively into all mathematics classrooms, we must begin by taking a serious look at our purpose. Schoenfeld gives six assumptions on the nature of mathematics and the nature of humans as learners.

1. The purpose of mathematics instruction is to help students think mathematically.
2. Mathematics is complex and highly structured at all levels.
3. Thinking mathematically includes mastering various facts and understanding the connections among them.
4. Thinking mathematically includes the ability to apply formal mathematics with knowledge, flexibility, and meaning.
5. Students are active interpreters of the world around them
6. These interpretations shape the way students see the world and use mathematical knowledge (164)

DEVELOPMENT

I would like to focus primarily on the first four assumptions from above. If the purpose of mathematical instruction is to help students think mathematically we cannot expect endless manipulations and contrived word problems to help us reach this goal. These methods of instruction leave no room for play or exploration. In addition it does not let students know that mathematics typically starts with a conjecture, proof, then a refutation (Sriraman 3). We need to give students opportunities to explore this process on their own. Sriraman suggests that all great mathematicians wrote their calculations, conjectures, results, etc in notebooks. Perhaps encouraging students to write their thought processes in journals would help them more easily make the transition to formal proof. In one of Sriraman’s classes he implemented “recreational” Diophantine problems to initiate the conjecture-proof-refutation idea among his students. After posing a problem, students would restate the problem, devise a strategy, solve the problem while writing about processes that worked and did not work, and finally what did you think about the problem. Students were encouraged to include all of their work in their journals. Full credit was given for following the cues not necessarily for a correct answer. It took months for students to become familiar with thinking about mathematics in this manner but it set the stage for successful future work with proof.

Secondly, according to Schoenfeld, thinking mathematically includes mastering various facts but more importantly, understanding the connections between them. Schoenfeld states that students do not use formal mathematics when doing problem
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solving. They do not make the necessary connections between proofs, problem solving, and constructions. Almeida suggests that students see proofs as an external activity instead of an internal activity meant to provide insight and understanding (871). In addition, Moore found that students had difficulty writing proof because they had little intuitive understanding of the concepts (251).

Another example of students’ inability to make connections can be found in Battista. He describes a series of interviews done by Schoenfeld where “college students were asked to use a straightedge and compass to construct the circle that is tangent to two intersecting lines with one point of tangency being a given point P on one of the lines.” The irony comes as one reads on to discover that during the same problem session, these same students had already proved that the center of a circle tangent to two given lines lies at the intersection of the angle bisector of the angle formed by the two lines and the perpendicular through the lines at the points of tangency. Despite this, 30% of students didn’t make the connection between the two. It seems that the proof activity did not help them understand the construction activity or vice versa. An artificially reconstructed logical proof conveys little insight to the student about the process or motivation for doing proof. We need to provide meaningful learning experiences for students. Unless we offer opportunities for engaging students in mathematical tasks that encourage them to want to look for proofs, then little progress in their mathematical understanding will be made.

Finally, thinking mathematically includes the ability to apply formal mathematics with knowledge, flexibility, and meaning. Students often need to develop a concept image through examples, diagrams, graphs, etc. before they can understand a formal
verbal definition or be able to make a proof (Moore 262). Providing students with experiences to work with informal proofs offers them opportunities to formulate conjectures. This may help them develop the inner drive to understand why a conjecture is true (Knuth 77). In addition, using a process oriented approach may make proof more accessible to students by allowing them to progress through stages of reasoning as well as providing investigations different from the typical content related targets (Hoyles 19). In many cases the transition to formal mathematics and proof is rather abrupt. Students need numerous opportunities to work with the concepts at a less formal level so that they may begin to work with their mathematical knowledge flexibly and with meaning.

APPLICATION

Despite the obvious importance of proof, NCTM states that teachers are not prepared to teach proof. Including proof effectively in the mathematics curriculum will require tremendous effort of the teachers (Knuth). Many beginning teachers today are products of the period of time when proof was not emphasized. Teachers feel much pressure to move on to the work, typically interpreted to be the exercises (Andrews 40). In a typical mathematics classroom students very rarely engage in a proof on their own. Math is always presented to them as a neat package without any struggles. This method of presentation encourages the students to think if you understand the material you can work the exercises. If you cannot work the problem you do not understand the material (Schoenfeld 159). If students have to spend more than five minutes on a problem, they do not think it is possible. It takes tremendous effort on the part of the teachers to teach proof in a way that foster students’ autonomy, gives students the ability to create their own knowledge, and creates independent thinkers (Knuth). In a rush to cover the
material teachers do not allow students time to reflect on what they are doing. If there is not time for reflection, then time on task is wasted (Andrews 40).

I believe there are many ways to apply proof in the classes I teach. I would like to look more closely at Battista’s use of Diophantine problems to initiate conjecture-proof-refutation. Diophantus is known for his work on solving equations with rational number solutions. The beauty of his problems is that they are easy to understand but are very difficult to solve. Many number theory problems can be posed as Diophantine equations giving algebra students an opportunity to gain experience with conjecture, proof and refutation.

By introducing Diophantine equations in a beginning algebra course as recreational journal problems students can begin to feel they have control over mathematics. Battista began with the classic n-tuple problem. A diophantine n-tuple is a set of n positive integers such that the product of any two is one less than a square integer. By starting with a simple version of the problem, the students eventually began to investigate 5-tuples. Many students became so excited about solving the problems many of them wrote algorithms or computer programs to look for integer solutions. As students formed conjectures the teacher could begin the process of proving their conjectures. The success of this method relied heavily on the will of the students. However, the students were willing and excited about doing mathematics. The most impressive part of this was the fact that fourteen year olds were capable of solving difficult problems, proving mathematics and powerful original thought. The use of journals allowed the teacher to gain insight and maintain constant communication with the students in order to encourage the students to prove these difficult problems.
In this example the teacher used one class period a week to have the students share ideas, present solutions, and discuss proofs. If the students had not worked on the problem throughout the week, the instructor simply went on with the planned algebra curriculum. The mathematics learned by the students through these “recreational” problems certainly enhanced their learning of the traditional algebra curriculum.

CONCLUSION

Proof is critical to the understanding of mathematics. It is important to have students begin informally proving mathematical concepts at a young age. This will increase their mathematical understanding as well as their connections between formal mathematics and problem solving. I believe that many times we discourage student’s discovery in mathematics by always showing them a prepackaged formula without showing them the difficulties that mathematicians went through to reach the conclusion. In addition, we rarely show students the struggles that we as instructors go through to solve problems. Allowing students to discover mathematics, struggle with it, and then prove their discoveries creates a much stronger mathematical environment as well as a much stronger mathematics student.

We must better prepare our teachers to teach proofs. They will need to have more meaningful experiences with conjecture-proof-refutation. By implementing “recreational” problems, and more real-life opportunities to use proof, we will make their learning experience stronger as well as the learning experience of their students. Proof is a powerful tool that is often overlooked in the rush to get through the exercises. We need to take the time to look deeply at the mathematics and allow our students the opportunity to truly understand mathematics.
REFERENCES


