The notion of proof has long played a key role in the study of mathematics. It is in my opinion the role of proof that separates mathematics from the sciences and other fields of study. It is the existence of proofs that give mathematicians the confidence that their work is credible and thus allows them to continue to build upon prior work without the need to second guess what has previously been accomplished.

Based upon this observation, it becomes natural to ask the questions pertaining to the use of proof in learning and understanding mathematics. If the concept of proof is so important to the field of mathematics, then is it possible that by writing proofs and studying proofs that an individual will be better equipped to understand the mathematics for which the proofs pertain? And if this is possible then when should a person be first exposed to proofs and at what level? In this paper I will give my views pertaining to these questions, as well as, a few more of my views pertaining to some other topics related to these questions.

Before discussing the virtues of proofs as a means of learning and understanding mathematics, I feel that it is first necessary to begin with a brief discussion of the functions of proof within mathematics. Following I will give a list of the functions of proof that I have comprised from three sources (Hanna [2], Knuth [3], Tucker [6]):

1. verification, the act of arguing that a statement is true
2. explanation, providing reasons for why a statement is true, which in turn may lead to understanding
3. systematization, organizing statements and definitions into a system of axioms, lemmas, theorems, etc.
4. discovery, creating knowledge and new results
5. communication, conveying mathematical knowledge between parties, whether individually or within a group
6. construction of an empirical theory (taken from Hanna)
7. exploration, to understand the meaning and/or consequences of a definition or statement
8. incorporation of a well known fact into a new framework (taken from Hanna)

This list is essentially taken from Hanna’s paper, but upon review of Knuth’s paper one would see that the first five functions are the same as the the conceptions of the role of proofs given by the teachers within his study. There are far less of these given in Tucker’s paper, but in order to show some continuity across the literature, I have included it because amongst his list of common reason’s why proofs should be used to teach mathematics is that “proofs help students understand concepts and believe results” (although he doesn’t share this view). Clearly this statement falls within items one and two, where “believe results” can be interpreted as verification and understanding goes along with the idea of explanation.

If we are to believe that this list of the functions of proof in mathematics has any credibility, then we should at least be tempted to believe that the use of proofs can play a vital role in learning and understanding mathematics. If proofs can provide a means of explanation, discovery, communication, and exploration; then wouldn’t the use of proofs be a powerful tool for learning mathematics? It is my opinion that the answer to this question is a resounding yes and consequently that means should be found for incorporating proofs into the mathematics curriculum.

Some important aspects involved during the process of learning mathematics are those of critical thinking and reasoning. The ability to reason and to think critically are both important tools for studying mathematics as well as valuable skills learned from the process of studying mathematics. In Fitzgerald’s paper [1], “Proof in Mathematics Education,” he gives the following observation:
Notice the action aspect of reasoning. It is the process of finding information from other information. Reasoning is not the mere memorizing of some information.

It is my belief that an important means for learning to reason (both mathematically and non-mathematically) is through the act of proving. When forming a proof a person is forced to work within the confines of the statement given by Fitzgerald. One can not simply form a mathematical proof by reciting some memorized facts. Furthermore, it should be clear that proof writing skills rely on “the process of finding information from other information.” Even if a person is not successful in writing a complete proof for a given problem the activity is not without merit. It is my belief that a person can still benefit from the work done while attempting to formulate a proof. I think that Thomas K. Tucker [6] stated this very well in his article, “On the Role of Proof in Calculus Courses,” when he stated:

the beauty of proof is much more likely to be appreciated when the questions in doubt and it is up to the student to grapple with the problem, even if unsuccessfully.

Throughout the paper “Proof, Explanation and Exploration: An Overview,” written by Gila Hanna [2], the usefulness of proofs for understanding mathematics is outlined. In one such instance he writes:

Through a closer examination of mathematical practice, I came to the further conclusion that even in the eyes of practicing mathematicians rigorous proof, however it is defined, is secondary in importance to understanding. It became clear to me that a proof, valid as it might be in terms of formal derivation, actually becomes both convincing and legitimate to a mathematician only when it leads to real mathematical understanding.
Although I find this statement a little extreme (I think a proof can be valuable as verification even if it doesn’t explain), I think it brings up an important point on the possible value of proofs as a means of understanding mathematics. In fact, I think it is safe to say that most mathematicians would prefer a proof that not only verifies a theorem, but also explains why it is true. Hanna furthers this discussion with the following statement:

*mathematicians clearly expect more of a proof than justification...they would also like it to make them wiser. This means that the best proof is one that also helps understand the meaning of the theorem being proved...Of course such a proof is also more convincing and more likely to lead to further discoveries.*

The final reason that I will give for the use of proofs for learning mathematics is that writing proofs can provide a means of exploration. It should be clear that exploration can be a useful tool for learning mathematics. In many classrooms students are given activities for exploring a new topic before a formal explanation is given. Also, it may be the activity of exploration that brings one the closest to actually experiencing the true nature of studying mathematics. And it is in my opinion that the act of writing a proof can lead to this exploration. I should point out that I am not advocating that exploration can only be accomplished through the act of forming a proof, but simply that it can be one of many tools for encouraging such behavior.

I am not alone in this view of proof as a means of exploration. In fact, I again find my self in agreement with Gila Hana based upon the following statement:

*while exploring and proving are separate activities, they are complementary and reinforce one another. Not only are they both part of problem solving in general, they are both needed for success in mathematics in particular. Exploration leads to discovery, while proof is confirmation.*
Now that I have attempted to establish the usefulness of proof in learning mathematics, I need to discuss the issue of how and when proofs should be introduced into a mathematics curriculum. In Frank K. Lester’s [4] paper, “Developmental Aspects of Children’s Ability to Understand Mathematical Proof,” he outlines some of the work of Piaget and others pertaining to the stages of understanding for children of various ages. The following is a brief outline of the highlights of this topic:

*The work of Piaget suggests that children pass through several stages of logical operations and problem-solving strategies. However, there is considerable evidence that some kinds of problem-solving processes are employed by children of all age levels.*

According to Piaget children of ages 11 to 13 years are able to handle certain formal operations successfully, but they are not able to set up an exhaustive method of proof. This ability to deal with premises that require hypothetico-deductive reasoning is not present until the child is approximately 14 to 15 years of age.

Children in the age range of six through eight years are able to recognize valid conclusions derived from hypothetical premises.

An examination of research involving the logical-reasoning abilities of young children reveals that these abilities may be far superior to their ability to put an argument in written form...This evidence suggests that the ability of children to create the essence of mathematical proofs may be superior to their ability to write proofs.

The reason that I have chosen for outlining Lester’s discussion is that the work above suggests that students of all ages are equipped at some level for dealing with
proofs or at least the concept of proof. I think that this may be an important realization. If the use of proofs can be incorporated in the mathematics curriculum at all levels then I think that it would have a greater chance for success in the aid of understanding and learning mathematics. Granted we may not be able to ask students of all ages to write mathematical proofs for a given topic, but should be able to ask them for reasons for which a topic may be true.

One means by which elements of proof can be introduced at any level is through the use of visual representations, such as diagrams. Mathematicians have long used visual representations as a means of portraying ideas that lead to proofs and it is my contention that students of all ages are capable of dealing with diagrams and using them to attempt to give proofs, whether the proof be written or oral. In the previously mentioned paper Gila Hanna touches on this topic:

*Diagrams and other visual aids have long been used to facilitate understanding, of course. They have been welcomed as heuristic accompaniments to proof, where they can inspire both the theorem to be proved and approaches to the proof itself. In this sense it is well accepted that a diagram is a legitimate component of a mathematical argument.*

Consequently, we can see that proof or at least elements of proof can be introduced at an early level. We may not be able to ask young children to write coherent proofs for a given theorem, but we can ask them to discuss mathematical phenomenon and to give reasons for their conclusions. In this way they will begin to build the tools necessary to create proofs, thus allowing them to develop the ability to reason and helping them to understand mathematics better.

At this time it is important to note that work has been done to determine how well students and teachers understand the role of proof. In the paper “Validations of Proofs Considered as Texts: Can Undergraduates Tell Whether an Argument Proves a Theorem?,” Annie and John Selden [5] attempt to determine whether students can
distinguish between a valid and a flawed proof; and in the paper “Secondary School Mathematics Teachers Conceptions of Proof,” Eric Knuth [3] attempts to determine secondary school teacher’s perceptions of proof. In both papers one can see some mixed results. Furthermore, in Eric J. Knuth’s paper he states:

> proof is expected to play a much more prominent role throughout the entire school mathematics curriculum and to be a part of the mathematics education of all students. Successfully enacting these new recommendations, however, places significant demands on school mathematics teachers because approaches designed to enhance the role of proof in the classroom require effort on their part. The challenge of meeting these demands is particularly daunting in light of the fact that many students find the study of proof difficult.

It is clear that both students and teachers sometimes have difficulty understanding the roles of proofs in mathematics, and although I think a remedy needs to be found for the teachers’ understandings, I would not take this as a negative indicator for the use of proofs throughout the curriculum. As I have outlined earlier the act of finding a proof can be a useful tool for exploration and understanding and for this reason I believe that the benefits of using proofs will outweigh the misperceptions of what a proof accomplishes mathematically and should also outweigh the difficulties in including proof within the curriculum.

This then leads us to the question of how do accomplish the goal of using proofs to teach mathematics. The first item that I think needs to be dealt with before accomplishing this goal is that of teacher understanding of the use and importance of proof within mathematics. As Knuth’s paper shows, there is some concern for how mathematics teachers view the use of proofs within mathematics. This needs to be remedied before we can expect teachers to convey to students the importance of proofs. Knuth touches on this with the following statement:
In short, teachers need, as students, to experience proof as a meaningful tool for studying and learning mathematics. Experiences of this nature may influence the conceptions of proof that they develop as teachers, and these ideas, in turn, may influence the experiences with proof their students will encounter in secondary school mathematics classrooms.

So by first changing teachers perceptions about proofs they will be better equipped to pass on the idea of the importance of proofs in mathematics to their students.

Once this is accomplished, teachers need to determine when and where proofs can be introduced into the classroom. If we want to use proofs within the classroom to accomplish the goals of explanation, discovery, communication, and exploration then we need to chose problems leading to proofs that accomplish at least one of these goals. I also believe, however, that the presentation of proofs as verification of a theorem can also be useful, but for the purpose of proofs as exercises I would keep to the four goals listed above.

Finally, I would like to give one last reason for why the use of proofs while learning mathematics might be useful. Possibly surprisingly this reason has little to do with mathematics, but might be useful across the curriculum. By requiring students to learn to create and write proofs they are accomplishing more than learning and obtaining understanding of mathematics. It is my opinion that the act of forming and writing mathematical proofs also aids in the ability to produce coherent and understandable arguments and that it can also help students to develop their writing skills. If this is the case then it should be clear that writing proofs can have many benefits to students that will lead to future success in many areas. Somewhat along these lines I will finish with this statement from Fitzgerald’s paper:

*Synthesis or proof is next to the highest form of cognitive development according to Bloom. Not to prove is to accept any statement in life without questioning its appropriateness. We do want to make students inquisitive.*
And I will add, we should also ourselves strive to stay inquisitive.
Bibliography


