1. Scalar product and Orthogonality.

- 1. In case of vector spaces, the scalar product \((A,B)\) is the dot product 
   \[ A \cdot B = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n, \]
   where \(A = [a_1, a_2, \ldots, a_n]^t; B = [b_1, b_2, \ldots, b_n]^t. \)

- 2. In case of function spaces \(\mathcal{F}\) over a given interval \([a, b]\), the scalar product defined as 
   \[ (f, g) = \int_a^b f(x)g(x)dx \]
   for any \(f, g \in \mathcal{F}.\)

- 3. Properties and requirements for scalar products:
   - Symmetry \((f, g) = (g, f)\).
   - Linearity \((f + g, h) = (f, h) + (g, h); (\alpha f, g) = \alpha (f, g) = (f, \alpha g)\).
   - Positivity \((f, f) > 0\) for all \(f \neq 0.\)

- 4. Length \(\|f\| = \sqrt{(f, f)}\). The angle \(\theta\) between \(f\) and \(g\) is represented as
   \[ \cos \theta = \frac{(f, g)}{\|f\| \cdot \|g\|} \]

- 5. Two elements \(f, g\) (vectors or functions) are said to be orthogonal (perpendicular, \(f \perp g\)), if \((f, g) = 0.\) (p.318,#4, 9,10),p.346#10(note that in the space \(\mathcal{F}[[-1,1],\perp x, x \perp x^2,\) but \(1 \perp x^2).\)

2. Projection.

- 1. Let \(Q_1, Q_2, \ldots, Q_m\) be a given orthogonal set, and let \(W = \text{span}\{Q_1, Q_2, \ldots, Q_m\}\)
   then for any element \(B\) (in or not in \(W\), its proj \(B_0\) on \(W\) is an element \(B_0 \in W\) such
   that the distance \(\|B - B_0\| = \text{minimum dist} \\|B - X\|\) for all \(X \in W.\)

- 2. Property. \((B - B_0) \perp W.\)

- 3. Find the projection-Fourier Theorem.
   In case of vector space \(R^n\): \(B_0 = \frac{(B, Q_1)}{\|Q_1\|^2} Q_1 + \frac{(B, Q_2)}{\|Q_2\|^2} Q_2 + \cdots + \frac{(B, Q_m)}{\|Q_m\|^2} Q_m\)

   In case of function space: \(f_0 = \frac{(f, q_1)}{\|q_1\|^2} q_1 + \frac{(f, q_2)}{\|q_2\|^2} q_2 + \cdots + \frac{(f, q_m)}{\|q_m\|^2} q_m\)

   where \(q_1, q_2, \ldots, q_m\) form an orthogonal set, and the scalar product is defined as \((f, g) = \int_a^b f(x)g(x)dx.\)

- 4. Fourier series. Especially, when \(q_1, q_2, \ldots, q_n\) form an orthonormal set, that is they
   are mutually orthogonal and all length = 1: \(\|q_k\| = 1, (k = 1, 2, \ldots, m), \)
   then \(f_0 = (f, q_1)q_1 + (f, q_2)q_2 + \cdots + (f, q_m)q_m\)

   where \(q_1, q_2, \ldots, q_m\)

- 5. Example: \(q_k(x) = \sin k \pi x, \cos k \pi x\) in the space \(\mathcal{F}([-1,1]).\) (p.332#5, p.334 #5.)

Note. The formulae given above about projections \(B_0\) and \(f_0\) are NOT true if \(Q_1, \ldots, Q_m\) or
\(q_1, \ldots, q_m\) do not form an orthogonal set. See items below.

3. Gram-Schmidt process.

Let \(A_1, A_2, \cdots A_m\) be independent vectors in \(R^n\) (if \(f_1, f_2, \cdots, f_m\) be indep in a function
spaces). To find an orthogonal set(p.332#6, p.346#10):
\[
\begin{align*}
Q_1 &= A_1 \\
Q_2 &= A_2 - \frac{(A_2, Q_1)}{||Q_1||^2} Q_1 \\
&\quad \quad \quad \vdots \quad \quad \quad \quad \quad \quad \vdots \\
Q_m &= A_m - \left( \frac{(A_m, Q_1)}{||Q_1||^2} Q_1 + \frac{(A_m, Q_2)}{||Q_2||^2} Q_2 + \cdots + \frac{(A_m, Q_{m-1})}{||Q_{m-1}||^2} Q_{m-1} \right)
\end{align*}
\]

4. Orthogonal subspace $S^\perp$. Let $S$ be a subspace (e.g., column space of a given matrix), the orthogonal subspace $S^\perp = \{ X : X \perp \text{all elements in } S \}$. The $S^\perp$ can be obtained by solving the equations $X \cdot P_i = 0$ where $P_i, i = 1, 2, \cdots, n$ are the elements of a basis of the subspace $S$. (p.333#14, 15.)

5. Orthogonal Matrices

- **1. Definition.** A matrix $A$ is said to be orthogonal if the length $\|Ax\| = \|x\|$ for all $x \in \text{Domain of } A$.

- **2. Characterization Theorem.** The matrix $A$ is orthogonal if and only if one of the following holds true:
  1. (i). Its column vectors from an orthonormal set.
  2. (ii). $A^t A = I$, namely, the transpose matrix $A^t$ is the inverse matrix of $A$.
(p359,#3,5,9,10,13).

6. Least square Theorem. Let $A \in M(m,n)$. When the vector $B \in R^m$ is not in the column space $S$ of the matrix $A$, the system $AX = B$ has no solution. But it can be approximately “solved” in the sense that the “solution” $X_0$ makes the distance $\|AX_0 - B\| = \min_{X \in S} \|AX - B\|$. The “solution” $X_0$ turns out to be the multiplication of matrices $X_0 = (A^t A)^{-1} A^t B$. Actually, the $B_0 := AX_0$ is the projection of the vector $B$ on the column space $S = \text{span}\{A_1, A_2, \cdots, A_n\}$. (Warning: Since $A$ may not be a square matrix, there may not be inverse $A^{-1}$. However, for $A \in M(m,n)$, the $A^t A$ is a $n$ by $n$ square matrix, and is invertible provided that rank$(A) = n$.)

The matrix $P = A(A^t A)^{-1} A^t$ is therefore termed as projection matrix. It has the property that $P^2 = P$.
(p.371,#2,6).