QUIZ 3

Show your work in detail for full credits.

1. Let matrix $A \in M(m, n)$. If $X$ is a solution to the system $AX = B$.
   (a) What is the number of components of vector $X$? $= n$
   (b) What is the number of components of vector $B$? $= m$
   (c) In order to have a solution $X$ to the system $AX = B$, which linear space must $B$ belong to?
      $B \in S_C$, the Column Space.
   (d) If vector $X$ is a solution to above system, suppose $Y \neq 0$ and $Y \in N(A)$, the null space.
      Show that the vector $X + Y$ is another solution to the above system.
      \[ A(X + Y) = A + AY = A0 + 0 = A0 = B \]
      So $A(X + Y) = B$ holds, i.e. $X + Y$ is a solution to the given system.
   (e) In order that the system $AX = B$ has at most one solution, what is the $\text{rank}(A)$?
      $\text{rank}(A) \geq n$
      (Because we need $\dim(N(A)) = 0$.
      \[ \dim(N(A)) = 0 \]
      \[ \dim(V) + \dim(N(A)) = n \]

2. Let $A = \begin{bmatrix} 2 & 1 & 3 & 1 \\ 1 & 1 & 3 & 0 \\ 0 & 1 & 2 & 1 \\ 3 & 3 & 8 & 2 \end{bmatrix}$.
   (a) Find independent column vectors of matrix $A$.
      The pivot columns in $\text{ref}(A)$ are 1st, 2nd, and 3rd columns.
      Two independent columns in $A$ are $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$.
   (b) Find a basis of column space $S_C$.
      Same as the 3 vcts for $\text{ref}(A)$:
      $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$.
   (c) Find a basis of the row space $S_R$.
      The pivot columns in $\text{ref}(A)$ (not the $A^T$):
      $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 5 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \\ 0 \\ -2 \end{bmatrix}$.
   (d) Find the independent row vectors.
   Perform the transpose matrix,
   \[ A^T = \begin{bmatrix} 2 & 1 & 0 & 3 \\ 1 & 1 & 1 & 3 \\ 3 & 3 & 8 & 2 \\ 1 & 8 & 1 & 2 \end{bmatrix} \]
   \[ \text{ref}(A^T) = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \]
   The 1st three columns of $\text{ref}(A^T)$ are pivots.
   Thus, the 1st three columns of $A^T$, i.e., the 1st three rows of $A$ are independent,
   $\begin{bmatrix} 2 \\ 1 \\ 3 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 3 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix}$.
(e) Find a basis of the null space $N(A)$.

Let $\text{Aug} = [A | 0] = \begin{bmatrix} 2 & 1 & 3 & 1 & | & 0 \\ 0 & 1 & 2 & 1 & | & 0 \\ 3 & 3 & 8 & 2 & | & 0 \end{bmatrix}$
\[ \text{Ref} (\text{Aug}) = \begin{bmatrix} 1 & 0 & 0 & 1 & | & 0 \\ 0 & 1 & 0 & 5 & | & 0 \\ 0 & 0 & 1 & 2 & | & 0 \end{bmatrix} \]

Let $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$.

$(NA), x, y, z$ are pivot variables thus $w$ is a free variable.

\[ \begin{align*}
    x + w &= 0 \\
    y + 5w &= 0 \\
    z - 2w &= 0
\end{align*} \]

Solve: $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = w \begin{bmatrix} -1 \\ -5 \\ 2 \\ 1 \end{bmatrix}$

$N(\mathbf{A}) = \text{Span}\left\{ \begin{bmatrix} -1 \\ -5 \\ 2 \\ 1 \end{bmatrix} \right\}$

(f) Let $\mathbf{B} = [1, 2, 3, 4]^T$. Does the system $\mathbf{A}\mathbf{x} = \mathbf{B}$ have solution(s)? If it does, find the solution(s). State the reason, if it doesn't.

Let $\text{Aug} = [A | B] = \begin{bmatrix} 2 & 1 & 3 & 1 & 6 \\ 0 & 1 & 2 & 1 & 3 \\ 3 & 3 & 8 & 2 & 4 \end{bmatrix}$
\[ \text{Ref} (\text{Aug}) = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 5 & 0 \\ 0 & 0 & 1 & 2 & 0 \end{bmatrix} \]

is inconsistent.

Therefore $\mathbf{A}\mathbf{x} = \mathbf{B}$ has no solution.

(g) Answer the same question as above for $\mathbf{B} = [6, 5, 3, 14]^T$.

Let $\text{Aug} = [A | B] = \begin{bmatrix} 2 & 1 & 3 & 1 & 6 \\ 0 & 1 & 2 & 1 & 3 \\ 3 & 3 & 8 & 2 & 4 \end{bmatrix}$, $\text{Ref} (\text{Aug}) = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 5 & 1 \\ 0 & 0 & 1 & 2 & 1 \end{bmatrix}$

Pivot variables: $x, y, z$.

Free $w, u$.

\[ \begin{align*}
    x + w &= 1 \\
    y + 5w &= 1 \\
    z - 2w &= 1
\end{align*} \]

Solve: $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \\ w \\ u \end{bmatrix} = \begin{bmatrix} 1 - w \\ 1 - 5w \\ 1 + 2w \\ w \\ u \end{bmatrix}$

Translation: $\mathbf{v}_0 = \begin{bmatrix} 1 \\ 8 \\ 0 \end{bmatrix}$, Spanning vector: $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

$\mathbf{x} = \begin{bmatrix} 1 \\ 8 \\ 0 \end{bmatrix} + \frac{1}{3} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.