Show your work in the space provided under each question.

1. (10 pts.) For each equation below answer the following questions: Is the equation linear or non-linear? Is it homogeneous or non-homogeneous? What is its order?

   (a) \( u_{xx} + u_{yy} = f(x, y) \)

   linear, non-homogeneous, 2nd order

   (b) \( u_t + x(u^3)_x = 0 \)

   nonlinear, homogeneous, 1st order

2. (10 pts.) Solve the equation \( \frac{\partial u}{\partial x} + 7 \frac{\partial u}{\partial t} = 0, \; u(x, 0) = f(x) \).

   \( u(x, t) = f(x - 7t) \)

   Then \( \frac{\partial u}{\partial x} + 7 \frac{\partial u}{\partial t} = f'(x - 7t) + 7(-7)f'(x - 7t) = 0 \)

   if \( \beta = \frac{1}{7} \)

   \( u(x, t) = f(x - \frac{1}{7}t) \)

3. (5 pts.) Draw graphs of the even and odd periodic extensions of the function. Include several periods.

   \( f(x) = x^2 \) on \( 0 \leq x \leq 1 \)

   \( \text{even extension} \)

   \( \text{odd extension} \)
4. (25 pts.) Show how to solve the following differential equation. You will need to use separation of variables. Explain all your steps.

\[ u_{tt} + u = u_{xx} \quad 0 \leq x \leq L, \quad t \geq 0 \]
\[ u(0, t) = 0 \quad t \geq 0 \]
\[ u(L, t) = 0 \quad t \geq 0 \]
\[ u(x, 0) = f(x) \quad 0 \leq x \leq L \]
\[ u_t(x, 0) = g(x) \quad 0 \leq x \leq L \]

Let \( u(x, t) = X(x)T(t) \). Plug in to get

\[ X T' + X T = X^{\prime\prime} T \]

\[ \frac{T^{\prime\prime}}{T} + 1 = \frac{X^{\prime\prime}}{X} \]

So both sides const., call it \( k \).

1. \( k > 0 \). Have \( \frac{X^{\prime\prime}}{X} = k = \lambda^2 \)
\[ X^{\prime\prime} - \lambda^2 X = 0 \]
\[ X(x) = c_1 e^{\lambda x} + c_2 e^{-\lambda x} \]
\[ 0 = X(0) = c_1 + c_2 \]
\[ 0 = X(L) = c_1 e^{\lambda L} + c_2 e^{-\lambda L} \]

So \( c_1 = -c_2 \)
\[ 0 = -c_2 (e^{\lambda L} + e^{-\lambda L}) \]
\[ \Rightarrow c_2 = 0 \text{ and so } c_1 = 0 \text{ no solutions} \]

2. \( k = 0 \)
\[ X^{\prime\prime} = 0 \]
\[ X(x) = cx + d \]
\[ 0 = X(0) = d \Rightarrow X(x) = cx \]
\[ 0 = X(L) = cL \Rightarrow c = 0 \]
no solutions
3. \( k < 0 \), write \( k = -\mu^2 \)

get \( x'' + \mu^2 x = 0 \)

\( x(x) = c_1 \cos \mu x + c_2 \sin \mu x \)

\( 0 = x(0) = c_2 \)

so \( x(x) = c_2 \sin \mu x \)

\( 0 = x(L) = c_2 \sin \mu L \)

need \( \mu L = n \pi \)

\( \mu = \frac{n \pi}{L} \)

get \( x_n(x) = c_n \sin \frac{n \pi x}{L} \)

Corresponding is \( \frac{T''}{T} + 1 = k = -\mu^2 = -\left(\frac{n \pi}{L}\right)^2 \)

\( T'' + \left(1 + \left(\frac{n \pi}{L}\right)^2\right) T = 0 \)

write \( \lambda_n = \sqrt{1 + \left(\frac{n \pi}{L}\right)^2} \)

\( T'' + \lambda_n^2 T = 0 \)

\( T(x) = c_n \cos \lambda_n x + d_n \sin \lambda_n x \)

get solutions \( u_{nL}(x,t) = \sin \frac{n \pi x}{L} \left( c_n \cos \lambda_n t + d_n \sin \lambda_n t \right) \)

\( u(x,t) = \sum_{n=0}^{\infty} \sin \frac{n \pi x}{L} \left( c_n \cos \lambda_n t + d_n \sin \lambda_n t \right) \)
\[ f(x) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{L} \]

So \[ c_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \, dx \]

\[ \frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{L} \left( c_n \frac{\partial}{\partial x} \left( \sin \frac{n\pi x}{L} \right) + a_n \frac{\partial}{\partial x} \cos \frac{n\pi x}{L} \right) \]

\[ g(x) = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{L} \, a_n \]

So \[ a_n = \frac{2}{\pi n} \int_0^L g(x) \sin \frac{n\pi x}{L} \, dx \]
5. (20 pts.) Find the solution to the heat equation

\[
\frac{\partial u}{\partial t} = 16 \frac{\partial^2 u}{\partial x^2} \quad t \geq 0, \quad 0 \leq x \leq 5
\]

\[
u(0, t) = u(5, t) = 0 \quad t \geq 0
\]

\[
u(x, 0) = x
\]

Solution is \( u(x, t) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{5} e^{-\frac{n^2 \pi^2 t}{25}} \)

Where \( a_n = \frac{2}{5} \int_{0}^{5} x \sin \frac{n\pi x}{5} \, dx \)

\[
a_n = \frac{2}{5} \left[ \left. x \cos \frac{n\pi x}{5} \right|_{0}^{5} - \int_{0}^{5} \cos \frac{n\pi x}{5} \, dx \right]
\]

\[
a_n = \frac{2}{5} \left[ \left. 5 \cos \frac{n\pi x}{5} \right|_{0}^{5} - \frac{\sin \frac{n\pi x}{5}}{\left(\frac{n\pi}{5}\right)^2} \right|_{0}^{5}
\]

\[
a_n = \frac{2}{5} \cdot \frac{4}{n\pi} (-\cos n\pi) = -\frac{2}{n\pi} (-1)^n \Rightarrow \frac{10}{n\pi} (-1)^{n+1}
\]

So solution is \( u(x, t) = \sum_{n=1}^{\infty} \frac{10}{n\pi} (-1)^{n+1} \sin \frac{n\pi x}{5} e^{-\frac{n^2 \pi^2 t}{25}} \)

\[
T_n = \frac{10}{n\pi} (-1)^{n+1} \]

\[
\frac{n\pi}{5}
\]
6. (20 pts.) Using the solution of d’Alembert, write out the solution of the wave equation

\[
\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \quad t \geq 0, \ 0 \leq x \leq 3
\]
\[
u(0, t) = u(3, t) = 0 \quad t \geq 0
\]
\[
u(x, 0) = f(x) \quad 0 \leq x \leq 3
\]
\[
\frac{\partial u}{\partial t}(x, 0) = 0 \quad 0 \leq x \leq 3
\]

where \( f \) is given by

\[
f(x) = \begin{cases} 
  x - 1 & \text{if } 1 \leq x \leq \frac{3}{2} \\
  2 - x & \text{if } \frac{3}{2} \leq x \leq 2
\end{cases}
\]

Then sketch the graphs of \( u(x, 0) \), \( u(x, 1) \) and \( u(x, 2) \).

Let \( \tilde{f} \) be the odd periodic extension of \( f \).

Then the solution is

\[
u(x, t) = \frac{1}{2} \left[ \tilde{f}(x+t) + \tilde{f}(x-t) \right]
\]
7. (10 pts.) Consider the heat equation

\[
\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad t \geq 0, \; 0 \leq x \leq L
\]

Show that at steady state the amount of heat entering the ends of the wire is zero, regardless of what boundary conditions are imposed. In other words, consider \( v(x) \), a steady state solution, and show that \( \frac{\partial v}{\partial x}(0) = \frac{\partial v}{\partial x}(L) \), regardless of the boundary conditions. Hint: The proof is very short.

\[ v \text{ satisfies } \frac{\partial^2 v}{\partial x^2} = 0, \]

so

\[
\frac{\partial v(L)}{\partial x} - \frac{\partial v(0)}{\partial x} = \int_0^L \frac{\partial^2 v}{\partial x^2} \, dx = 0
\]