Show your work in the space provided under each question. There are 150 points possible.

1. (10 pts.) For each equation below answer the following questions: Is the equation linear or non-linear? Is it homogeneous or non-homogeneous? What is its order?
   
   (a) \( u_{xx} + (\cos u)_x = 0 \)
   
   *non-linear, homogeneous, second order*

   (b) \( u_t + xu_x^2 = e^t \)
   
   *non-linear, non-homogeneous, first order*

2. (10 pts.) Draw the graph of the even and odd extensions of the function. Include several periods.

   \( f(x) = x^3 \) on \( 0 \leq x \leq 1. \)

   ![Graph of even and odd extensions of \( f(x) = x^3 \).]
3. (20 pts.) Show how to solve the following differential equation. You will need to use separation of variables. Explain all your steps.

\[ u_{tt} + 4u = u_{xx} \quad 0 \leq x \leq L, \quad t \geq 0 \]
\[ u(0, t) = 0 \quad t \geq 0 \]
\[ u(L, t) = 0 \quad t \geq 0 \]
\[ u(x, 0) = f(x) \quad 0 \leq x \leq L \]
\[ u_t(x, 0) = g(x) \quad 0 \leq x \leq L \]

\[ U = XT \quad XT'' + 4XT = X''T \]

Divide by XT
\[ \frac{T''}{T} + 4 = \frac{X''}{X} \]

Both sides constant, call it k.

1. \( k > 0 \). Write \( k = \mu^2 \) \[ \frac{X''}{X} = \mu^2 \]
\[ X'' - \mu^2 X = 0 \]
\[ X = c_1 e^{\mu x} + c_2 e^{-\mu x} \]

Need \( 0 = u(0, t) = X(0)T(t) \). So need \( 0 = X(0) = c_1 + c_2 \)

Also need \( 0 = u(L, t) = X(L)T(t) \). So need \( 0 = X(L) = c_1 e^{\mu L} + c_2 e^{-\mu L} \)

\[ c_1 + c_2 = 0 \]
\[ c_1 e^{\mu L} + c_2 e^{-\mu L} = 0 \]
\[ \Rightarrow c_1 = c_2 = 0 \quad \text{no solns} \]

2. \( k = 0 \)
\[ X'' = 0 \]
\[ X = c_1 x + c_2 \]
\[ 0 = X(0) \Rightarrow c_2 = 0 \]
\[ 0 = X(L) \Rightarrow c_1 = 0 \]
\[ \text{no solns} \]

3. \( k < 0 \). Write \( k = -\mu^2 \)
\[ \frac{X''}{X} = -\mu^2 \]
\[ X'' + \mu^2 X = 0 \]

Which gives \( X(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x) \)

\[ 0 = X(0) = c_1 \]
\[ 0 = X(L) = c_2 \sin(\mu L) \quad \text{so need } \mu L = n\pi, \]
\[ n = 1, 2, 3, \ldots \quad \text{so } \mu = \frac{n\pi}{L} \]

So get solns \( X_n = \sin\left(\frac{n\pi}{L} x\right) \). Find corresponding T.
\[ \frac{T''}{T} + 4 = -\mu^2 = -\left(\frac{n\pi}{L}\right)^2 \]
\[ \Rightarrow T'' + \left(\frac{n\pi}{L}\right)^2 + 4)T = 0 \]

This gives \( T(t) = c_1 \cos\left(\left(\frac{n\pi}{L}\right)^2 + 4\right) t + c_2 \sin\left(\left(\frac{n\pi}{L}\right)^2 + 4\right) t \)

\[ U = XT \quad XT'' + 4XT = X''T \]

Divide by XT
\[ \frac{T''}{T} + 4 = \frac{X''}{X} \]

Both sides constant, call it k.

1. \( k > 0 \). Write \( k = \mu^2 \) \[ \frac{X''}{X} = \mu^2 \]
\[ X'' - \mu^2 X = 0 \]
\[ X = c_1 e^{\mu x} + c_2 e^{-\mu x} \]

Need \( 0 = u(0, t) = X(0)T(t) \). So need \( 0 = X(0) = c_1 + c_2 \)

Also need \( 0 = u(L, t) = X(L)T(t) \). So need \( 0 = X(L) = c_1 e^{\mu L} + c_2 e^{-\mu L} \)

\[ c_1 + c_2 = 0 \]
\[ c_1 e^{\mu L} + c_2 e^{-\mu L} = 0 \]
\[ \Rightarrow c_1 = c_2 = 0 \quad \text{no solns} \]

2. \( k = 0 \)
\[ X'' = 0 \]
\[ X = c_1 x + c_2 \]
\[ 0 = X(0) \Rightarrow c_2 = 0 \]
\[ 0 = X(L) \Rightarrow c_1 = 0 \]
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3. \( k < 0 \). Write \( k = -\mu^2 \)
\[ \frac{X''}{X} = -\mu^2 \]
\[ X'' + \mu^2 X = 0 \]

Which gives \( X(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x) \)

\[ 0 = X(0) = c_1 \]
\[ 0 = X(L) = c_2 \sin(\mu L) \quad \text{so need } \mu L = n\pi, \]
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So get solns \( X_n = \sin\left(\frac{n\pi}{L} x\right) \). Find corresponding T.
\[ \frac{T''}{T} + 4 = -\mu^2 = -\left(\frac{n\pi}{L}\right)^2 \]
\[ \Rightarrow T'' + \left(\frac{n\pi}{L}\right)^2 + 4)T = 0 \]

This gives \( T(t) = c_1 \cos\left(\left(\frac{n\pi}{L}\right)^2 + 4\right) t + c_2 \sin\left(\left(\frac{n\pi}{L}\right)^2 + 4\right) t \)
Write \( \lambda_n = \left( \frac{\pi n}{L} \right)^2 + 4 \)

So

\[
U(x, t) = \sum_{n=0}^{\infty} (a_n \cos \lambda_n t + b_n \sin \lambda_n t) \sin \left( \frac{\pi n}{L} x \right)
\]

want

\[
f(x) = u(x, 0) = \sum_{n=0}^{\infty} a_n \sin \left( \frac{\pi n}{L} x \right)
\]

Take \( a_n = \frac{2}{L} \int_0^L f(x) \sin \left( \frac{\pi n}{L} x \right) \, dx \)

\[
\frac{\partial u(x, t)}{\partial x} = \sum_{n=0}^{\infty} (-n^2 \lambda_n \sin \lambda_n t + n \lambda_n b_n \cos \lambda_n t) \sin \left( \frac{\pi n}{L} x \right)
\]

want

\[
g(x) = \frac{\partial u(x, 0)}{\partial t} = \sum_{n=0}^{\infty} n \lambda_n b_n \sin \left( \frac{\pi n}{L} x \right)
\]

need \( b_n = \frac{1}{\lambda_n} \frac{2}{L} \int_0^L g(x) \sin \left( \frac{\pi n}{L} x \right) \, dx \)
4. (15 pts.) Find the solution to the Dirichlet problem:

\[ \Delta u(r, \theta) = 0 \quad 0 \leq r < 1, \ 0 \leq \theta \leq 2\pi \]
\[ u(1, \theta) = 10 \quad 0 \leq \theta \leq \pi \]
\[ u(1, \theta) = -10 \quad \pi < \theta < 2\pi \]

\[
U(r, \theta) = a_0 + \sum_{n=1}^{\infty} r^n \left[ a_n \cos n\theta + b_n \sin n\theta \right]
\]

\[
a_0 = \frac{1}{2\pi} \int_{0}^{2\pi} u(1, \theta) d\theta = \frac{1}{2\pi} \int_{0}^{\pi} 10 \, d\theta + \frac{1}{2\pi} \int_{\pi}^{2\pi} -10 \, d\theta = 0
\]

\[
a_n = \frac{1}{\pi} \int_{0}^{\pi} u(1, \theta) \cos n\theta \, d\theta
\]

\[
b_n = \frac{1}{\pi} \int_{0}^{\pi} u(1, \theta) \sin n\theta \, d\theta
\]

\[
U(r, \theta) = \sum_{n=1}^{\infty} \frac{10}{n\pi} \left(2 - (2n-1)^n\right) \sin n\theta
\]

\[
U(r, \theta) = \sum_{n=1}^{\infty} \frac{40}{n\pi\sin n\theta} \frac{1}{2} n^{\text{odd}}
\]

\[
U(r, \theta) = \sum_{n=1}^{\infty} \frac{40}{n\pi\sin n\theta} \frac{1}{2} n^{\text{even}}
\]
5. (15 pts.) Use Fourier transforms to solve

\[
\frac{\partial u}{\partial x} + 3 \frac{\partial u}{\partial t} = 0 \quad u(x, 0) = f(x)
\]

on \(-\infty < x < \infty, \ t > 0\).

\[
+i \omega \hat{u}(\omega, t) + 3 \frac{\partial \hat{u}}{\partial t}(\omega, t) = 0 \quad \hat{u}(\omega, 0) = \hat{f}(\omega)
\]

\[
3 \frac{\partial \hat{u}}{\partial t} = -i \omega \hat{u}
\]

\[
\frac{\partial \hat{u}}{\hat{u}} = -\frac{1}{3} i \omega
\]

\[
l(\omega) = -\frac{1}{3} i \omega t + \mathcal{L}
\]

\[
\hat{u}(\omega, t) = C e^{-\frac{1}{3} i \omega t}
\]

\[
t = 0 \quad \hat{u}(\omega) = \hat{u}(\omega, 0) = \mathcal{L}
\]

\[
\hat{u}(\omega, t) = \hat{f}(\omega) e^{-\frac{1}{3} i \omega t}
\]

so \(u(x, t) = \frac{1}{\sqrt{\pi t}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{\frac{i}{3} \omega t} \cdot e^{-i \omega x} \ d\omega
\]

\[
= \hat{f}(x - \frac{t}{3})
\]

\[
= f(x - \frac{t}{3})
\]
6. (20 pts.) Consider the inhomogeneous heat equation

\[
\frac{\partial u}{\partial t}(x, t) = \frac{\partial}{\partial x} \left( K(x) \frac{\partial u}{\partial x} \right) \quad t \geq 0, \ 0 \leq x \leq L
\]

\[
u(0, t) = 0 \quad t \geq 0
\]

\[
u(L, t) = 0 \quad t \geq 0
\]

\[
u(x, 0) = f(x)
\]

Solve the initial value problem using separation of variables. You will not be able to get an explicit solution, but only as a series involving the eigenfunctions of a Sturm-Liouville problem.

\[
u = XT
\]

\[
\begin{align*}
XT_1 &= \frac{\partial}{\partial x}(KX_1T) \\
XT_1 &= T \frac{\partial}{\partial x}(KX_1) \\
\frac{T}{T} &= \frac{\partial}{\partial x}(KX_1) \quad \text{so both sides const., call it } k.
\end{align*}
\]

1. \(k > 0\). Write \(k = m^2\), then get \(\frac{T}{T} = m^2, T' - m^2 T = 0\)

\[
e^{-m^2 t} T - m^2 T e^{-m^2 t} = 0
\]

\[
\frac{dT}{dt} (e^{-m^2 t} T) = 0
\]

\[
e^{-m^2 t} T = C
\]

\[
T = Ce^{m^2 t}
\]

Unrealistic \(T(t) \to 0\).

2. \(k = 0\). \(\frac{T}{T} = 0, T' = 0 \Rightarrow T(t) = C t\). Unrealistic either \(C > 0\) and \(T(t) = 0\) or \(C < 0\) then \(T(t) \to 0\) or \(C < 0\) then \(T(t) \to -\infty\).

3. \(k < 0\). Write \(k = -m^2\). Get \(\frac{T}{T} = -m^2, \text{ so } T' + m^2 T = 0\)

\[
T(t) = C e^{-m^2 t}
\]
find corresponding $x$.

These satisfy $-m^2 = \frac{\partial^2 (kx')}{\partial x^2}$.

Also $0 = u(0, t) = x(0)T(t)$, Need $x(0) = 0$

and $0 = \frac{\partial u}{\partial t}(L, t) = x'(L)T(t)$, Need $x'(L) = 0$

Write $-m^2 = -\lambda (\lambda > 0)$.

Get the Sturm-Liouville problem

$\frac{\partial^2 (kx')}{\partial x^2} + \lambda x = 0$.

$x(0) = 0$

$x'(L) = 0$.

This has eigenvalues $\lambda_1, \lambda_2, \ldots$ and corresponding eigenfunctions $\phi_1, \phi_2, \ldots$

Set $U(x, t) = \sum_{n=1}^{\infty} a_n \phi_n(x) e^{-\lambda_n t}$

Want $f(x) = u(x, 0) = \sum_{n=1}^{\infty} a_n \phi_n(x)$. Take $a_n = \frac{1}{\int_0^L [\phi_n(x)]^2 dx} \int_0^L \phi_n(x) f(x) dx$
7. (15 pts.) Consider a circular drum of radius 1. Suppose the drum is set into motion with an initial displacement of \( f(r) = r(1 - r) \) and no initial velocity. Write the boundary value problem for the function \( u(r, \theta, t) \) which gives the displacement of the drum from equilibrium at the point \((r, \theta)\) at time \(t\). Then solve the problem, writing your answer as a series involving the Bessel function \( J_0 \). You should indicate how to compute the coefficients of the series, but you do not need to attempt to compute the integrals for these.

\[
\begin{align*}
\Delta u + \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} &= 0 \\
u(1, \theta, t) &= 0, \quad \theta \in (-\pi, \pi) \\
u(r, \theta, 0) &= r(1 - r), \quad 0 \leq r \leq 1, \quad \theta \in (-\pi, \pi) \\
\frac{\partial \nu}{\partial t}(r, \theta, 0) &= 0
\end{align*}
\]

So, the solution is

\[
u(r, t) = \sum_{n=1}^{\infty} \left( a_n \cos(n \omega t) + b_n \sin(n \omega t) \right) J_0 (\lambda_n r)
\]

where \( \lambda_n = \frac{\alpha_n}{1} = n \beta \) zero on \( J_0 \).

\[
a_n = \frac{2}{J_1 (\alpha_n)} \int_0^1 r(1-r) J_0 (\lambda_n r) r dr
\]

\[
b_n = 0
\]
8. (15 pts.) Determine the eigenvalues and eigenfunctions of the given Sturm-Liouville problem.

\[ y'' + \lambda y = 0, \quad y(0) = 0, \quad y(\pi) + y'(\pi) = 0. \]

1. \( \lambda < 0 \). Write \( T = -\mu^2 \), \( y'' - \mu^2 y = 0 \)

\[ y = c_1 e^{\mu x} + c_2 e^{-\mu x} \]

need \( 0 = y(0) = c_1 + c_2 \)

Also \( y' = c_1 \mu e^{\mu x} - \mu c_2 e^{-\mu x} \)

so need \( 0 = y(\pi) + y'(\pi) = c_1 e^{\mu \pi} + c_2 e^{-\mu \pi} + c_1 \mu e^{\mu \pi} - \mu c_2 e^{-\mu \pi} \)

\( = c_1 (1 + \mu) e^{\mu \pi} + c_2 (1 - \mu) e^{-\mu \pi} \)

\( c_1 + c_2 = 0 \Rightarrow c_2 = -c_1. \) Plug into last eqn and get

\( c_1 (1 + \mu) e^{\mu \pi} - c_1 (1 - \mu) e^{-\mu \pi} = 0 \)

\( (1 + \mu) e^{\mu \pi} = (1 - \mu) e^{-\mu \pi} \)

\( e^{2\mu \pi} = \frac{1 - \mu}{1 + \mu} \)

impossible. Left is \( > 1 \), right is < 1

2. \( \lambda = 0 \). \( y'' = 0 \), \( y(x) = c_1 x + c_2, \ y(0) = 0 \Rightarrow c_2 = 0 \)

\( y'(x) = c_1 \), \( y'(0) = c_1 \). need \( c_1 + c_1 = 0 \Rightarrow c_1 = 0 \)

3. \( \lambda > 0 \). Write \( T = \mu^2 \) \( y = c_1 \cos \mu x + c_2 \sin \mu x \)

\( 0 = y(0) = c_1. \) So \( y = c_2 \sin \mu x \). Then \( y' = \mu c_2 \cos \mu x \)

Need \( 0 = y(\pi) + y'(\pi) = c_2 \sin \mu \pi + \mu c_2 \cos \mu \pi \)

Need \( \frac{\sin \mu \pi}{\cos \mu \pi} = -\mu \) i.e. \( \tan \mu \pi = -\mu \)

e.vals are solns \( \mu_n \) of this

\( y = \sin \mu_n x \)
9. (15 pts.) On the range \(-\infty < x < \infty, \ t > 0\), determine the solution of the heat equation

\[
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2},
\]

\[u(x, 0) = 100, \text{ if } -1 \leq x \leq 1, \ u(x, 0) = 0, \text{ otherwise}\]

\[
U(x, t) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} f(s) e^{-\frac{(x-s)^2}{4t}} \, ds
\]

\[
= \frac{100}{2\sqrt{\pi t}} \int_{-1}^{1} e^{-\frac{(x-s)^2}{4t}} \, ds
\]
10. (15 pts.) Consider the heat equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad t \geq 0, \, 0 \leq x \leq a, \, 0 \leq y \leq b.$$ 

Show that at steady state the amount of heat entering the rectangle is zero, regardless of what boundary conditions are imposed. In other words, consider \( v(x, y) \), a steady state solution, and show that the net flux

$$\int_0^a \frac{\partial v}{\partial y}(x, 0) \, dx + \int_0^b \frac{\partial v}{\partial x}(a, y) \, dy + \int_0^a \frac{\partial v}{\partial y}(x, b) \, dx + \int_0^b \frac{\partial v}{\partial x}(0, y) \, dy$$

is zero regardless of the boundary conditions. Hint: Write down the equation that \( v \) satisfies and integrate this over the rectangle. Then use the fundamental theorem of calculus.

At steady state, \( \Delta v = 0 \)

i.e. \( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \)

So

$$0 = \int_0^a \int_0^b \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \, dx \, dy$$

$$= \int_0^b \int_0^a \frac{\partial^2 v}{\partial x^2} \, dx \, dy + \int_0^a \int_0^b \frac{\partial^2 v}{\partial y^2} \, dy \, dx$$

$$= \int_0^b \left( \frac{\partial v}{\partial x}(a, y) - \frac{\partial v}{\partial x}(0, y) \right) \, dy + \int_0^a \left( \frac{\partial v}{\partial y}(x, b) - \frac{\partial v}{\partial y}(x, 0) \right) \, dx$$

$$= \int_0^a -\frac{\partial v}{\partial y}(x, 0) \, dx + \int_0^b \frac{\partial v}{\partial x}(a, y) \, dy + \int_0^a \frac{\partial v}{\partial y}(x, b) \, dx$$

$$+ \int_0^b -\frac{\partial v}{\partial x}(0, y) \, dy$$