PROBLEM 1. Let \( T : \mathbb{R}^2 \rightarrow \mathbb{R}^3 \) be the linear transformation given by

\[
T(x_1, x_2) = \begin{pmatrix}
2x_1 + x_2 \\
3x_1 \\
4x_2 
\end{pmatrix}
\]

(i) Find a \( 3 \times 2 \) matrix \( A \) such that \( Tx = Ax \) for every \( x = (x_1, x_2) \in \mathbb{R}^2 \).

Solution. The first thing to remember is that every linear transformation \( T \) can be represented by a certain matrix \( A \). We construct the matrix \( A \) as follows: Since \( T \) is taking values from \( \mathbb{R}^2 \), we consider the canonical basis for \( \mathbb{R}^2 \) which is

\[
B = \{ [1,0]', [0,1]' \}
\]

We usually denote these vectors by \( e_1 \) and \( e_2 \). That is, \( e_1 = [1,0]' \) and \( e_2 = [0,1]' \). Now, the desired matrix \( A \) is the one that has \( T(e_1) \) and \( T(e_2) \) as columns. In our case

\[
T(e_1) = T(1,0) = \begin{pmatrix}
2 \\
3 \\
0
\end{pmatrix}
\]

and

\[
T(e_2) = T(0,1) = \begin{pmatrix}
1 \\
0 \\
4
\end{pmatrix}
\]

Therefore, the matrix \( A \) that represents \( T \) is

\[
A = \begin{pmatrix}
2 & 1 \\
3 & 0 \\
0 & 4
\end{pmatrix}
\]

(ii) Find a basis for \( \text{range}(T) \). Is \( T \) onto?

Solution. By definition, we have that \( \text{range}(T) = \text{col}(A) \) where \( \text{col}(A) \) is the column space of \( A \), that is, the subspace generated by the columns of \( A \). Then, we only need to find a basis for \( \text{col}(A) \). Clearly, the vectors \( [2,3,0]' \) and \( [1,0,4]' \) generate the column space of \( A \) (because they are precisely the columns of \( A \)). To determine whether they form a basis or not we have to answer the following question: Are those vectors linearly independent? Clearly, they are, since one is not a multiple of the other. Then a basis for \( \text{col}(A) \) is

\[
\{ [2,3,0]', [1,0,4]' \}
\]
By definition, $T$ is onto if $\text{range}(T)$ equals all the target space, in our case, $\mathbb{R}^3$. Thus, the question is: Is it true that $\text{range}(T) = \mathbb{R}^3$?

Clearly the answer is no. This is so because we just saw that $\{(2, 3, 0)', (1, 0, 4)\}'$ is a basis for $\text{range}(T) = \text{col}(A)$. If we had $\text{range}(T) = \mathbb{R}^3$ then this would imply that $\{(2, 3, 0)', (1, 0, 4)\}'$ is also a basis for $\mathbb{R}^3$. This is impossible since $\mathbb{R}^3$ has dimension 3 and $\{(2, 3, 0)', (1, 0, 4)\}'$ has only 2 vectors. (With just two linearly independent vectors one cannot generate a three-dimensional space, one just generates a plane)

(iii) Let $b = [0 \ 0 \ 0]'$. If we have that

$$>> \text{rref}([A \ b])$$

ans =

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

What is $\ker(T)$? Is $T$ one-to-one?

**Solution.** By definition, $\ker(T)$ is the collection of $x = (x_1, x_2)$ such that $Tx = 0$. Using the representation matrix $A$, we get that $x = (x_1, x_2)$ belongs in the kernel of $T$ if $Ax = 0$. This immediately takes us to the system

$$Ax = \begin{pmatrix}
2 & 1 \\
3 & 0 \\
0 & 4
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix} = \begin{pmatrix}
0 \\
0
\end{pmatrix}$$

We solve the system by getting the reduced row echelon form of the augmented matrix $[A \ b]$, which is given above. Translating back into a system we get

$$\begin{pmatrix}
1 & 0 \\
0 & 1 \\
0 & 0
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix} = \begin{pmatrix}
0 \\
0
\end{pmatrix}$$

That is $x_1 = x_2 = 0$. Therefore, $\ker(T) = \{(0, 0)\}'$, that is, $\ker(T)$ is the zero subspace which (by Theorem 3.20) means exactly that $T$ is in fact one-to-one.

**PROBLEM 2.** Find $x$ and $y$ such that the matrix $Q$ given by

$$Q = \begin{pmatrix}
x & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \\
\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & y
\end{pmatrix}$$

is orthogonal.

**Solution.** By Theorem 4.6 of the textbook, we know that $Q$ being orthogonal means that its columns are orthonormal vectors, that is, its columns are orthogonal to each other and they all have norm equal to 1.

Take the first column, for instance, since that column is supposed to have norm 1, we have

$$x^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 = 1.$$
Thus, \( x^2 + 1 = 1 \) and hence \( x = 0 \).

On the other hand, the third row is also supposed to have length one too. Then,
\[
\left( \frac{1}{\sqrt{3}} \right)^2 + \left( -\frac{1}{\sqrt{3}} \right)^2 + y^2 = 1.
\]
That is, \( y^2 = 1 - 2/3 = 1/3 \), which gives \( y = \pm \sqrt{1/3} \). In order to determine whether we should choose the plus or minus sign we use the orthogonality. It is clear that if we chose the minus sign, then the third column would not be orthogonal to the first one or to the second one (the corresponding dot products wouldn’t be zero). Then \( y = \sqrt{1/3} \), and everything works like a charm.

**PROBLEM 3.** Find the projection of the vector \( v = [1, 0, -1, 4]' \) onto the subspace spanned by the vectors \( u_1 = [1, 1, 1, 1]' \) and \( u_2 = [1, -1, 1, -1]' \).

**Solution.** This is another piece of cake problem. By Theorem 4.4 of the textbook (the one we implemented using MATLAB) we know that, since \( u_1 \) and \( u_2 \) are orthogonal (check it), the desired projection \( \overline{v} \) is given by
\[
\overline{v} = p_1 u_1 + p_2 u_2,
\]
where \( p_1 = \frac{v \cdot u_1}{\|u_1\|^2} \) and \( p_2 = \frac{v \cdot u_2}{\|u_2\|^2} \). We then have
\[
p_1 = \frac{v \cdot u_1}{\|u_1\|^2} = \frac{1 + 0 - 1 + 4}{1^2 + 1^2 + 1^2 + 1^2} = \frac{4}{4} = 1
\]
and
\[
p_2 = \frac{v \cdot u_2}{\|u_2\|^2} = \frac{1 + 0 - 1 - 4}{1^2 + (-1)^2 + 1^2 + (-1)^2} = \frac{-4}{4} = -1
\]
Finally,
\[
\overline{v} = p_1 u_1 + p_2 u_2 = u_1 - u_2 = [0, 2, 0, 2]
\]

**PROBLEM 4.** Use cross products to find the area of the triangle on the plane whose vertices are \( P = (1, 1), Q = (3, 5) \) and \( R = (6, 4) \).

**Solution.** This is problem is very useful. The first difficulty we encounter is that the cross product is defined on vectors of \( \mathbb{R}^3 \), not on vectors of \( \mathbb{R}^2 \), and our points are on the plane \( \mathbb{R}^2 \). This is easily fixed that considering that everything takes place in \( \mathbb{R}^3 \). Think that the plane \( \mathbb{R}^2 \) is just the “floor” of \( \mathbb{R}^3 \). By doing this, our points become \( P = (1, 1, 0), Q = (3, 5, 0) \) and \( R = (6, 4, 0) \). That is, their third coordinate (the “height”) is zero. Now we construct two vectors that form the sides of our triangle, say
\[
u = Q - P = (3, 5, 0) - (1, 1, 0) = [2, 4, 0]'\]
and
\[
v = R - P = (6, 4, 0) - (1, 1, 0) = [5, 3, 0]'\]
By one of the properties of the cross product we know that \( \|u \times v\| \) is the area of the parallelogram that has sides \( u \) and \( v \). The area we are looking for is half of that. We can easily compute the cross product and get
\[
u \times v = [0, 0, -14]'.
\]
This vector has norm equal to 14, so the desired area is 7.

**PROBLEM 5.** Suppose that we have three square matrices \( A, B, \) and \( C \) such that \( A \) is invertible and \( AB = CA^T \). Here \( A^T \) is the transpose of \( A \). Find \( \det(C) \) assuming that \( \det(B) = 4 \). Is \( C \) invertible?

**Solution.** Recall that the determinant of a product is the product of the determinants and that the determinant of a matrix equals the determinant of its transpose. Since \( AB = CA^T \), we get
\[
\det(AB) = \det(CA^T)
\]
But \( \det(AB) = \det(A) \det(B) \), and \( \det(CA^T) = \det(C) \det(A^T) = \det(C) \det(A) \). Hence
\[
\det(A) \det(B) = \det(C) \det(A).
\]
Since we know that \( A \) is invertible we have that \( \det(A) \) is different from zero, and then we can divide by \( \det(A) \) in the equality above to get
\[
\det(B) = \det(C).
\]
Hence, \( \det(C) = \det(B) = 4 \) (by hypothesis). In particular, since \( \det(C) \) is different from zero, we deduce that \( C \) is invertible. Finally, since the determinant of the inverse is the inverse of the determinant we do
\[
\det(C^{-1}) = (\det(C))^{-1} = 4^{-1} = 1/4.
\]

**PROBLEM 6** Is \( T \) linear?

(i) **YES - NO.** \( T(x_1, x_2) = x_1 + 2x_2 + 3 \).

**NO.** For instance, we have that \((1, 1) = (1, 0) + (0, 1)\) but \( T(1, 1) = 6 \) is not \( T(1, 0) + T(0, 1) = 4 + 5 = 9 \).

(ii) **YES - NO.** \( T(x_1, x_2, x_3) = [x_1 - x_2, 3x_1 + 2x_2]' \).

**YES.** This \( T \) is \( T_A \) where \( A \) is the representing matrix given by
\[
\begin{pmatrix}
1 & -1 & 0 \\
3 & 2 & 0
\end{pmatrix}
\]

Is \( U \) subspace?
(i) YES - NO. \( U = \{ x = [x_1, x_2, x_3]' \in \mathbb{R}^3 : x_1 + x_3 = 0 \} \).

**YES.** \( U \) is exactly null(\( A \)) where \( A = [1 \ 0 \ 1] \).

(ii) YES - NO. \( U = \{ x = [x_1, x_2]' \in \mathbb{R}^2 : x_2 = 2x_1 \} \).

**YES.** \( U \) is exactly null(\( A \)) where \( A = [-2 \ 1] \).

**PROBLEM 7.** In the following cases explain why the matrix \( A \) in question is diagonalizable or not.

Case 1.

\[
\begin{align*}
&[P \ D] = \text{eig}(A) \\
&P = [0 \ 0 \ 0.4841; 0 \ 0.1961 -0.8713; 1.0000 -0.9806 \ 0.0807] \\
&D = [7 \ 0 \ 0; 0 \ 6 \ 0; 0 \ 0 \ 1]
\end{align*}
\]

What is \( \text{det}(A) \)?

**Solution Case 1.** The matrix IS diagonalizable because all its eigenvalues are distinct (no repeated eigenvalues in the diagonal of \( D \)). \( \text{det}(A) = \text{det}(D) = 7 \times 6 \times 1 = 42 \).

Case 2.

\[
\begin{align*}
&[P \ D] = \text{eig}(A) \\
&P = [0 - 0.7071i \ 0 + 0.7071i \ 0; 0.7071 \ 0.7071 \ 0; 0 \ 0 \ 1.0000] \\
&D = [1.0000 + 5.0000i \ 0 \ 0; 0 \ 1.0000 - 5.0000i \ 0; 0 \ 0 \ 1.0000]
\end{align*}
\]

**Solution Case 2.** The matrix is NOT diagonalizable in the field of real numbers since its eigenvalues are complex numbers.

Case 3.

\[
\begin{align*}
&[P \ D] = \text{eig}(A) \\
&P = [0 \ 0.7071 \ 0.7071; 0 \ 0 \ 0.7071; 1.0000 \ 0.7071 \ 0] \\
&D = [1 \ 0 \ 0; \ 0 \ -1 \ 0; \ 0 \ 0 \ 1]
\end{align*}
\]

**Solution Case 3.** To check that \( A \) is diagonalizable we only need to determine whether the matrix \( P \) is invertible or not. \( A \) is diagonalizable when the matrix \( P \) is invertible. That is, if its columns are linearly independent. It is readily seen that they actually are linearly independent (just take a look at them), so the matrix \( A \) IS diagonalizable.

Case 4.

\[
\begin{align*}
&[P \ D] = \text{eig}(A) \\
&P = [-0.7071 \ 0.7071; 0.7071 \ 0.7071] \\
&D = [0 \ 0; 0 \ 2]
\end{align*}
\]

**Solution Case 4.** To check that \( A \) is diagonalizable we only need to determine whether the
matrix $P$ is invertible or not. $A$ is diagonalizable when the matrix $P$ is invertible. That is, if its columns are linearly independent. It is readily seen that they actually are linearly independent (just take a look at them, one is not a multiple of the other), so the matrix $A$ IS diagonalizable. Here you can also assert that $A$ is diagonalizable because its eigenvalues are distinct.

Case 5.

$$\begin{bmatrix} P & D \end{bmatrix} = \text{eig}(A)$$

$$P = \begin{bmatrix} 1.0000 & 1.0000; 0 0.0000 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 0; 0 1 \end{bmatrix}$$

**Solution Case 5.** To check that $A$ is diagonalizable we only need to determine whether the matrix $P$ is invertible or not. $A$ is diagonalizable when the matrix $P$ is invertible. That is, if its columns are linearly independent. It is readily seen that they are NOT linearly independent (they are exactly the same), so the matrix $A$ is NOT diagonalizable.