1. State the quadratic formula and state when it applies!

If \( ax^2 + bx + c = 0 \), and \( a \neq 0 \), then

\[
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.
\]

A few comments are in order. (You don’t need to write these on the answer to the question above on the test.)

(a) Often quadratic polynomials can be factored and then you don’t need to use the quadratic formula.

(b) You can “peel off” the formula for the axis of symmetry of a parabola by realizing that the zeros are located at

\[
\frac{-b}{2a} + D \quad \text{and} \quad \frac{-b}{2a} - D
\]

with \( D \) defined appropriately.

(c) There is a nice connection between geometry and algebra here when considering the graph of the parabola \( y = ax^2 + bx + c \). Namely, this graph will hit the x-axis twice (geometry!) if \( b^2 - 4ac > 0 \) (algebra!), the graph touches the x-axis once if \( b^2 - 4ac = 0 \), and it does not touch if \( b^2 - 4ac < 0 \). All of this follows by understanding and applying the quadratic formula correctly!

(d) **Do not miss an equal sign! Do not miss an \( x \)!</p>
3. Solve for $x$ and $y$:

(a) $2x + y = 7$

(b) $2x - 3y = 1$

(c) $4x + 3y = 2$

$x - y = -1$

$3x + 2y = 17$

$2x + 9y = -1$

(a) By simply adding the equations together we get $3x = 6$. From there it is easy to get $x = 2$ and then $y = 3$.

(b) Multiplying the top equation by 3 and the bottom equation by $-2$ gives the equations:

$$6x - 9y = 3$$

$$-6x - 4y = -34$$

Adding together gives $-13y = -31$, so $y = \frac{31}{13} = 2 \frac{5}{13}$. Now you could proceed by plugging into one of the equations and solving for $x$, but I think it is easier to start from scratch, and multiply the top equation (from the original problem!) by 2 and the bottom equation by 3. This multiplication gives us:

$$4x - 6y = 2$$

$$9x + 6y = 51$$

Adding together we get $13x = 53$, so $x = \frac{53}{13} = 4 \frac{1}{13}$.

4. Find $f(g(x))$, $g(f(x))$, and $f(x)g(x)$ if

(a) $f(x) = 2x$, $g(x) = x - 5$.

$$f(g(x)) = 2g(x) = 2(x - 5) = 2x - 10.$$  

$$g(f(x)) = f(x) - 5 = 2x - 5.$$  

$$f(x)g(x) = 2x(x - 5) = 2x^2 - 5x.$$  

(b) $f(x) = 3x + 2$, $g(x) = -2x + 5$.

$$f(g(x)) = 3(g(x)) + 2 = 3(-2x+5)+2 = -6x+15+2 = -6x+17.$$
\[ g(f(x)) = -2(f(x)) + 5 = -2(3x + 2) + 5 = -6x - 4 + 5 = -6x + 1. \]

\[ f(x)g(x) = (3x + 2)(-2x + 5) \]
\[ = -6x^2 + 15x - 4x + 10 \]
\[ = -6x^2 + 11x + 10. \]

(c) \( f(x) = 2x + 3, \quad g(x) = 2x^2 - 5x. \)

\[ f(g(x)) = 2g(x) + 3 = 2(2x^2 - 5x) + 3 = 4x^2 - 10x + 3. \]

\[ g(f(x)) = 2(f(x))^2 - 5(f(x)) \]
\[ = 2(2x + 3)^2 - 5(2x + 3) \]
\[ = 2(2x + 3)(2x + 3) - 5(2x + 3) \]
\[ = 2(4x^2 + 12x + 9) - 5(2x + 3) \]
\[ = 8x^2 + 24x + 18 - 10x - 15 \]
\[ = 8x^2 + 14x + 3. \]

\[ f(x)g(x) = (2x + 3)(2x^2 - 5x) \]
\[ = 4x^3 - 10x^2 + 6x^2 - 15x \]
\[ = 4x^3 - 4x^2 - 15x. \]

(d) \( f(x) = \frac{2}{x - 1}, \quad g(x) = x^2 + 3. \)

\[ f(g(x)) = \frac{2}{g(x) - 1} = \frac{2}{x^2 + 3 - 1} = \frac{2}{x^2 + 2}. \]
\[ g(f(x)) = (f(x))^2 + 3 \]
\[ = \left( \frac{2}{x-1} \right)^2 + 3 \]
\[ = \left( \frac{2}{x-1} \right) \left( \frac{2}{x-1} \right) + 3 \]
\[ = \frac{4}{(x-1)^2} + 3 \quad \text{good enough here, but you should follow the rest.} \]
\[ = \frac{4}{(x-1)^2} + \frac{3(x-1)^2}{(x-1)^2} \]
\[ = \frac{4 + 3(x-1)^2}{(x-1)^2} \]
\[ = \frac{4 + 3(x-1)(x-1)}{(x-1)^2} \]
\[ = \frac{4 + 3(x^2 - 2x + 1)}{(x-1)^2} \]
\[ = \frac{3x^2 - 6x + 7}{(x-1)^2}. \]

\[ f(x)g(x) = \frac{2}{x-1}(x^2 + 3) = \frac{2x^2 + 6}{x-1}. \]

(e) \( f(x) = x^3, \ g(x) = x^4. \)

\[ f(g(x)) = (g(x))^3 = (x^4)^3 = x^{12}. \]
\[ g(f(x)) = (f(x))^4 = (x^3)^4 = x^{12}. \]
\[ f(x)g(x) = x^3x^4 = x^7. \]

5. Assume \( y = f(x). \) Find \( f^{-1}(y) \) if

(a) \( f(x) = 2x + 3. \)
\[
y = 2x + 3 \\
y - 3 = 2x \\
x = f^{-1}(y) = \frac{y - 3}{2}.
\]

(b) \( f(x) = \frac{3x - 2}{7} \).

\[
y = \frac{3x - 2}{7} \\
7y = 3x - 2 \\
7y + 2 = 3x \\
x = f^{-1}(y) = \frac{7y + 2}{3}.
\]

(c) \( f(x) = \frac{5}{3}(x - 32) \).

\[
y = \frac{5}{3}(x - 32) \\
\frac{9}{5}y = x - 32 \\
x = f^{-1}(y) = \frac{9}{5}y + 32.
\]

(d) \( f(x) = 8x^3 - 27 \).

\[
y = 8x^3 - 27 \\
y + 27 = 8x^3 \\
\frac{y + 27}{8} = x^3 \\
x = f^{-1}(y) = \left(\frac{y + 27}{8}\right)^\frac{1}{3} = \frac{3\sqrt[3]{y + 27}}{2}.
\]

(e) \( f(x) = \frac{3}{2x + 5} \).
\[
y = \frac{3}{2x + 5}
\]

\[
y(2x + 5) = 3
\]

\[
2x + 5 = \frac{3}{y}
\]

\[
2x = \frac{3}{y} - 5 = \frac{3y - 5y}{y}
\]

\[
2x = \frac{3 - 5y}{y}
\]

\[
x = f^{-1}(y) = \frac{3 - 5y}{2y}
\]

6. Word problems:

(a) The temperature, \( u \), of water in a pot on the stove \( t \) minutes after it is placed there is given by \( u = h(t) = 92 + 20t \) for \( 0 \leq t \leq 6 \). Find the time as a function of temperature.

\[
u = 92 + 20t
\]

\[
u - 92 = 20t
\]

\[
\frac{u - 92}{20} = t .
\]

(b) My distance, \( d \), from Manhattan on my trip to Austin, \( t \) hours after leaving is given by \( d = h(t) = 60t \) for \( 0 \leq t \leq 10 \). Find the time as a function of distance.

\[
t = \frac{d}{60} .
\]
7. True or False:

(a) The graph of every function satisfies the vertical line test.
   \textbf{True.} A function (say } y = f(x) \text{) is a rule which assigns exactly one value } y \text{ to any value } x. 

(b) The graph of every function satisfies the horizontal line test.
   \textbf{False.} The graph of every \textbf{invertible} function satisfies the horizontal line test.

(c) If } y = f(x) \text{ and } x = f^{-1}(y), \text{ then } x = f^{-1}(f(x)) \text{ for every } x \text{ in the domain of } f.
   \textbf{True,} by the definition of inverse function.

(d) If } y = f(x) \text{ and } x = f^{-1}(y), \text{ then } y = f(f^{-1}(y)) \text{ for every } y \text{ in the domain of } f^{-1}.
   \textbf{True,} by the definition of inverse function.

(e)
\[ f^{-1}(y) = \frac{1}{f(y)}. \]
   \textbf{False.} The inverse function is not (in general) the reciprocal of the original function.

(f)
\[ (f(y))^{-1} = \frac{1}{f(y)}. \]
   \textbf{True.} \( A^{-1} = \frac{1}{A} \). Note also the following convention which confuses many students:
\[ f^p(x) := (f(x))^p \text{ as long as } p \neq -1. \]
   Whenever \( f^{-1} \) is written, we are talking about an inverse function, not the function raised to the power \(-1\).

(g) If } y = f(g(x)) \text{ and both } f \text{ and } g \text{ are invertible, then }
\[ x = f^{-1}(g^{-1}(y)) \text{.} \]
   \textbf{False.}
\[ x = g^{-1}(f^{-1}(y)) \text{.} \]
(Think about opening boxes inside other boxes.)