Complex Exponentials, Sines, and Cosines.

Consider the equation $y'' + 6y' + 25y = 0$. Some students have asked questions about how we go from

$$y = C_1 e^{(-3+4i)x} + C_2 e^{(-3-4i)x}$$

to

$$y = k_1 e^{-3x} \cos(4x) + k_2 e^{-3x} \sin(4x).$$

The key is that to find the general solution, we just need to find any two different ("linearly independent") solutions and then put them together. As was noted by one of the students during lecture, this means we can write the general solution in many different forms. For example, if we choose $C_1 = i/2$, $C_2 = i/2$, we get

$$y = \frac{1}{2} e^{(-3+4i)x} + \frac{1}{2} e^{(-3-4i)x} = e^{-3x} \cos(4x)$$

and if we choose $C_1 = i$, $C_2 = -i/2$, we get

$$y = \frac{1}{2} i e^{(-3+4i)x} - \frac{1}{2} i e^{(-3-4i)x} = e^{-3x} \sin(4x),$$

so $e^{-3x} \cos(4x)$ and $e^{-3x} \sin(4x)$ are two different solutions of the homogeneous equation, so

$$y = k_1 e^{-3x} \cos(4x) + k_2 e^{-3x} \sin(4x)$$

is another form
of the general solution. To check the given values
for $c_1$ and $c_2$ produce $e^{-3x}\cos(4x)$ and $e^{-3x}\sin(4x)$,
you use Euler's formula $e^{ix} = \cos(x) + i\sin(x)$ along
with the fact that sine and cosine are odd and
even functions respectively (i.e., $-\sin(x) = \sin(-x)$
and $\cos(x) = \cos(-x)$). Discovering these identities
(written in terms of $\cosh(x)$ and $\sinh(x)$) was the
main part of the extra credit assignment over
Complex Variables.

Note that our two forms of the general solution,
\[ y = c_1 e^{(3+4i)x} + c_2 e^{(3-4i)x} \quad \text{and} \quad y = k_1 e^{-3x}\cos(4x) + k_2 e^{-3x}\sin(4x) \]
give exactly the same set of solutions to any
initial value problem. The second form is usually
much more convenient, because everything is real,
but you will get the same answer from the first form
if you do all the algebra correctly, as illustrated on the last page.
Solving an initial value problem the long way

\[ \frac{d^2y}{dx^2} + 6 \frac{dy}{dx} + 25y = 0 \]

\[ y(0) = 9, \quad y'(0) = -7 \]

\[ (D^2 + 6D + 20)y = 0 \]

Roots: \(-3 \pm 4i\)

\[ y = c_1 e^{(-3+4i)x} + c_2 e^{(-3-4i)x} \]

\[ y'(x) = (-3+4i)c_1 e^{(-3+4i)x} + (-3-4i)c_2 e^{(-3-4i)x} \]

\[ y(0) = c_1 + c_2 = 9 \]

\[ y'(0) = (-3+4i)c_1 + (3-4i)c_2 = -7 \] Multiply 1st equation by \(3-4i\)

Add these two equations

\[ -8ic_2 = 20 - 36i \]

\[ c_2 = \frac{20 - 36i}{-8i} = \frac{9}{2} + \frac{5}{2}i \]

Now \(c_1 + c_2 = 9\)

\[ c_1 + \frac{9}{2} + \frac{5}{2}i = 9 \]

\[ c_1 = \frac{9}{2} - \frac{5}{2}i \]

Finally

\[ y(x) = \left( \frac{9}{2} - \frac{5}{2}i \right) e^{(-3+4i)x} + \left( \frac{9}{2} + \frac{5}{2}i \right) e^{(-3-4i)x} \]

\[ = e^{-3x} \left[ \left( \frac{9}{2} - \frac{5}{2}i \right) \left( \cos(4x) + i\sin(4x) \right) \right. \]

\[ \left. + \left( \frac{9}{2} + \frac{5}{2}i \right) \left( \cos(4x) - i\sin(4x) \right) \right] \]

\[ = e^{-3x} \left[ \frac{9}{2} \cos 4x + \frac{9}{2} \sin 4x - \frac{5}{2} i \cos 4x + \frac{5}{2} i \sin 4x \right. \]

\[ + \frac{9}{2} \cos 4x - \frac{9}{2} i \sin 4x + \frac{5}{2} i \cos 4x + \frac{5}{2} i \sin 4x \]

\[ = e^{-3x} \left( 9\cos 4x + 5\sin 4x \right) \]