Show all your work in the space under each question. Please write legibly and organize your solutions in a logical and coherent form; answers which are illegible or confusing will not receive credit. Each problem is worth 10 points.

1. Find the general solution: \( y'' + 6y' + 3y = 0 \).

\[
D^2 + 6D + 3 = 0
\]

\[
D = \frac{-6 \pm \sqrt{36 - 4 \cdot 3}}{2} = \frac{-6 \pm \sqrt{36 - 12}}{2} = \frac{-6 \pm \sqrt{24}}{2} = \frac{-6 \pm 2\sqrt{6}}{2}
\]

\[
D = -3 \pm \sqrt{6}
\]

\[
y = c_1 e^{-3x} + c_2 e^{-\sqrt{6}x}
\]
2. The size of a population of bacteria (in thousands) at time $t$, $P(t)$, is given by the equation

$$\frac{dP}{dt} = -P(P - 2)(P - 10).$$

(a) What are the critical points of this equation?

$$P = 0, P = 2, P = 10$$

(b) Draw a graph of several trajectories of this equation. Be sure to include graphs of the equilibrium (constant) solutions, as well as a few trajectories between each of the equilibrium trajectories.

$$P < 0 \quad \frac{dP}{dt} = -P(P - 2)(P - 10) = -(-)(-)(-) > 0, \quad P \text{ inc}.$$  

$$0 < P < 2 \quad \frac{dP}{dt} = -P(P - 2)(P - 10) = -(+)(-)(-) < 0, \quad P \text{ dec}.$$  

$$2 < P < 10 \quad \frac{dP}{dt} = -P(P - 2)(P - 10) = -(+)(+)(-) < 0, \quad P \text{ inc}.$$  

$$P > 10 \quad \frac{dP}{dt} = -P(P - 2)(P - 10) = -(+) (+) (+) < 0, \quad P \text{ dec}.$$  

(c) It is observed that the population is currently 7,000. Do you expect the population to increase or decrease in the future? Explain your answer.

If $P(0) = 7$ then, from the graph,

$$\lim_{t \to \infty} P(t) = 10.$$  

Thus the population increases to 10,000.
3. Solve the initial value problem:

\[ y'' - 6y' + 9y = 0 \quad y(0) = 0, \quad y'(0) = 1. \]

\[ D^2 - 6D + 9 = 0 \]

\[ (D - 3)^2 = 0 \]

\[ y = c_1 e^{3x} + c_2 xe^{3x} \]

\[ 0 = y(0) = c_1 \cdot 1 + c_2 \cdot 0 \]

so \( c_1 = 0 \)

Thus, \( y = c_2 xe^{3x} \)

Then \( y' = c_2 e^{3x} + c_2 3x e^{3x} \)

\( 1 = y'(0) = c_2 + 0 \)

so \( c_2 = 1 \)

Then \( y = xe^{3x} \)
4. The solution of the initial value problem

\[
dP \over dt = P(P - 2) \quad P(0) = 3
\]

has a vertical asymptote at a finite time \( T \), that is, there is an explosion at some \( T < \infty \).
Find the solution to this initial value problem and find \( T \).

This is separable

\[
\frac{dP}{P(P - 2)} = dt
\]

\[
\frac{1}{P(P - 2)} = \frac{A}{P} + \frac{B}{P - 2}
\]

\[\frac{1}{P} = A + B \quad \Rightarrow \quad A = -\frac{1}{2}, \quad B = \frac{1}{2}\]

\( P = 2 \) gives \( B = \frac{1}{2} \), \( P = 0 \) gives \( A = -\frac{1}{2} \)

\[
\ln P \rightarrow \int -\frac{1}{2} \frac{1}{P} + \frac{1}{2} \frac{1}{P - 2} dP = \int dt
\]

\[
-\frac{1}{2} \ln |P| + \frac{1}{2} \ln |P - 2| = t + c
\]

\[
\frac{1}{2} \ln \left|\frac{P - 2}{P}\right| = t + c
\]

\[
P - \frac{2}{P} = ke^{2t}
\]

\[
P(0) = 3 \quad \frac{3 - 2}{3} = k \quad \Rightarrow \quad k = \frac{1}{3}
\]

\[
P - \frac{2}{P} = \frac{1}{3} e^{2t}
\]

\[
P - 2 = P \left(\frac{1}{3} e^{2t}\right)
\]

\[
P \left(1 - \frac{1}{3} e^{2t}\right) = 2
\]

\[
P = \frac{2}{1 - \frac{1}{3} e^{2t}}
\]

There is a vertical asymptote when \( 1 - \frac{1}{3} e^{2t} = 0 \), i.e.

\[
t = \log_{\frac{3}{2}} 3 \quad \Rightarrow \quad \text{explosion at } T = \log_{\frac{3}{2}} 3 \]
5. Use the Euler method with stepsize $h = .1$ to estimate $y(.2)$ for the initial value problem

$$y'' + y' + xy = 0 \quad y(0) = 0, \quad y'(0) = 1.$$  

Rewrite as a system. Let $z = y'$. Then the equation becomes

$$z' + z + xz = 0.$$  

Get system

$$y' = z \quad y(0) = 0$$  

$$z' = -z - xz \quad z(0) = 1$$  

Here $x_0 = 0 \quad x_1 = .1 \quad x_2 = .2 \quad y_0 = 0 \quad z_0 = 1$  

Then

$$y_1 = y_0 + hz_0 = 0 + .1(1) = .1$$  

$$z_1 = z_0 + h(-z_0 - x_0 y_0) = 1 + .1(-1 - 0) = .9$$  

$$y_2 = y_1 + hz_1 = .1 + .1(.9) = .19$$
6. Find the general solution: \( y''' - 6y'' + 11y' - 6y = 0 \).

\[
D^3 - 6D^2 + 11D - 6 = 0
\]

\( D = 2 \) is a root. \( x^3 - 6x^2 + 11x - 6 = (x - 2)^3 - 8x^2 + 22x - 6 = 0 \)

to find the other roots, we divide.

\[
\begin{array}{c|ccccc}
 & D^2 - 4D + 3 \\
\hline
D - 2 & D^3 - 6D^2 + 11D - 6 \\
 & D^3 - 2D^2 \\
 & -4D^2 + 11D \\
 & -4D^2 + 8D \\
 & 3D - 6 \\
 & 3D - 6
\end{array}
\]

so \( D^3 - 6D^2 + 11D - 6 = (D - 2)(D^2 - 4D + 3) = (D - 2)(D - 1)(D - 3) \)

roots are 1, 2, 3

so the solution is

\[
y = C_1e^x + C_2e^{2x} + C_3e^{3x}
\]
7. (a) Write in the form $A \cos(ax - \phi): \cos(2x) + \sqrt{3} \sin(2x)$.

\[
\cos 2x = \Re e^{i2x} \\
\sqrt{3} \sin 2x = \Re (-\sqrt{3} i e^{i2x})
\]

so \[
\cos 2x + \sqrt{3} \sin 2x = \Re (e^{i2x} + -\sqrt{3} i e^{i2x}) = \Re ((1-\sqrt{3} i) e^{i2x})
\]

rewrite \(1-\sqrt{3} i = \Re e^{i\theta}\)

\[
\sqrt{r^2 + \sqrt{3}^2} = 2 \\
\theta = -\frac{\pi}{3}
\]

so \(1-\sqrt{3} i = 2 e^{-\frac{\pi i}{3}}\)

thus continuing,

\[
\cos 2x + \sqrt{3} \sin 2x = \Re (2 e^{-\frac{\pi i}{3}} e^{i2x}) = \Re (2 e^{i(2x-\frac{\pi}{3})}) = 2 \cos(2x-\frac{\pi}{3})
\]

(b) Write in the form $a + bi: e^{\theta + \frac{\pi i}{3}}$.

\[
e^{\theta + \frac{\pi i}{3}} = e^{\frac{\pi i}{3}} = e^{\frac{\pi}{3}} (\cos\frac{\pi}{3} + i \sin\frac{\pi}{3})
\]

\[
= e^\theta (\frac{1}{2} + i \frac{\sqrt{3}}{2})
\]

\[
= \frac{e^\theta}{2} + i \frac{e^\theta \sqrt{3}}{2}
\]
8. Consider the autonomous equation \( \frac{dp}{dt} = \sin(2\pi p) \).

(a) What are the equilibrium points of the equation?

When \( \sin(2\pi p) = 0 \)

\[ 2\pi p = n\pi, \quad n \text{ an integer} \]
\[ p = \frac{n}{2}, \quad n \text{ an integer} \]

(b) Draw a graph of \( p \) vs. \( \frac{dp}{dt} \). (Put \( p \) on the horizontal axis, \( \frac{dp}{dt} \) on the vertical.)

(c) From your graph in (b), determine which of the equilibrium points are stable and which are unstable.

Where \( p = \pm \frac{1}{2}, \pm \frac{3}{2}, \pm \frac{5}{2}, \text{ etc} \) \( \text{ eq. pt. is stable} \)

When \( p = 0, \pm 1, \pm 2, \pm 3, \ldots \) \( \text{ eq. pt is unstable} \)
9. Find the general solution: $y'' + y' + y = 0$.

$$D^2 + D + 1 = 0$$

$$D = -1 \pm \frac{\sqrt{1 - 4}}{2}$$

$$D = -1 \pm \frac{\sqrt{3}}{2} i$$

$$D = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$$

So $y = c_1 e^{\frac{1}{2}x} \cos \frac{\sqrt{3}}{2} x + c_2 e^{\frac{1}{2}x} \sin \frac{\sqrt{3}}{2} x$
10. Find the general solution:

\[ y'' + y' = 1 + e^{2x} \]

\[
\begin{align*}
D^2 + 10 &= 0 & D &= 0 \\
D(D - 1) &= 0 & D &= -1 \\

y_h &= c_1 + c_2 e^{-x}
\end{align*}
\]

To find \( y_p \), find \( y_{p1} \) for \( y'' + y' = 1 \) and \( y_{p2} \) for \( y'' + y' = e^{2x} \).

For \( y_{p1} \), try \( y_{p1} = Ax \). Then \( y_{p1}' = A \) and \( y_{p1}'' = 0 \).

Feed in: \( 0 + A = 1 \)

So \( A = 1 \)

Hence \( y_{p1} = x \).

For \( y_{p2} \), try \( y_{p2} = Ae^{2x} \). Then \( y_{p2}' = 2Ae^{2x} \) and \( y_{p2}'' = 4Ae^{2x} \).

Feed in: \( 4Ae^{2x} + 2Ae^{2x} = e^{2x} \)

\[
\begin{align*}
6Ae^{2x} &= e^{2x} \\
A &= \frac{1}{6}
\end{align*}
\]

So \( y_{p2} = \frac{1}{6} e^{2x} \)

Thus \( y_p = x + \frac{1}{6} e^{2x} \)

So \( y = y_h + y_p = c_1 + c_2 e^{-x} + x + \frac{1}{6} e^{2x} \)