Show all your work in the space under each question. Please write legibly and organize your solutions in a logical and coherent form; answers which are illegible or confusing will not receive credit. Each problem is worth 10 points.

1. Solve.

\[ \frac{dy}{dx} + \frac{y}{x} = \log(x) \quad y(1) = 2. \]

It's linear. Integrating factor is \( e^{\int \frac{1}{x} dx} = e^{\log x} = x \)

\[ x\frac{dy}{dx} + y = x\log x \]

\[ \frac{d}{dx}(xy) = x\log x \]

\[ xy = \int x\log x \, dx = \frac{x^2}{2} \log x - \int \frac{x^2}{2} \, dx \]

\[ xy = \frac{x^2}{2} \log x - \frac{1}{4} x^2 + C \]

\[ y = \frac{x^2}{2} \log x - \frac{1}{4} x^2 + C \]

\[ +2 \quad y(1) = 2 \cdot 2 = \frac{1}{2} \log 1 - \frac{1}{4} + C \]

\[ y = \frac{1}{2} \log x - \frac{1}{4} x + \frac{C}{2} \]

Solution is

\[ y = \frac{x^2}{2} \log x - \frac{1}{4} x + \frac{C}{2} \]
2. Solve.

\[
\frac{dy}{dx} = \frac{x^2 + 2}{x \sin y}
\]

This is separable

\[
\sin y \, dy = \frac{x^2 + 2}{x} \, dx
\]

\[
\int \sin y \, dy = \int \frac{x^2 + 2}{x} \, dx
\]

\[
- \cos y = \int \left( x + \frac{2}{x} \right) \, dx
\]

\[
- \cos y = \frac{x^2}{2} + 2 \log |x| + C
\]

\[
\cos y = -\frac{x^2}{2} - 2 \log |x| + C
\]

\[
y = \arccos \left( -\frac{x^2}{2} - 2 \log |x| + C \right)
\]
3. Solve.

\[ 2x + y^2 + 2xy' = 0. \]

\[ \frac{\partial}{\partial y} (2x + y^2) = 2y \quad \frac{\partial}{\partial y} (2xy) = 2y \]

It's exact.

Solution is \( F(x, y) = C \) where \( \frac{\partial F}{\partial x} = 2x + y^2 \)

\[ \frac{\partial F}{\partial y} = 2xy \]

\[ \frac{\partial F}{\partial x} = 2x + y^2 \Rightarrow F(x, y) = x^2 + xy^2 + k(y) \]

Then \[ 2xy + k'(y) = \frac{\partial F}{\partial y} = 2xy \]

So \( k'(y) = 0 \)

Take \( k(y) = 0 \)

Then \( F(x, y) = x^2 + xy^2 \)

Solution to differential equation is then

\[ x^2 + xy^2 = C \]
4. Solve.

\[
\frac{dy}{dx} = \frac{x + y}{x - y}
\]

This is homogeneous.

Substitute \( y = xv \), \( x' = v + xv' \)

\[
v + xv' = \frac{x + xv}{x - xv}
\]

\[
v + xv' = \frac{1 + v}{1 - v}
\]

\[
xv' = \frac{1 + v}{1 - v} - \frac{v(1 - v)}{1 - v}
\]

\[
xv' = \frac{1 + v^2}{1 - v}
\]

\[
\frac{1 - v}{1 + v^2} \, dv = \frac{1}{x} \, dx
\]

\[
\int \frac{1 - v}{1 + v^2} \, dv = \int \frac{1}{x} \, dx
\]

\[
\arctan v - \frac{1}{2} \log(1 + v^2) = \log|x| + C
\]

\[
v = \frac{x}{2} \quad \arctan \left( \frac{x}{2} \right) - \frac{1}{2} \log(1 + (\frac{x}{2})^2) = \log|x| + C
\]
5. Solve.

\[ x \frac{dy}{dx} + 6y = 3xy^{\frac{4}{3}} \]

\[ \frac{dy}{dx} + \left( \frac{6}{x} \right)y = 3y^{\frac{4}{3}} \]

This is a Bernoulli equation. \( v = y^{1-\frac{4}{3}} = y^{-\frac{1}{3}} \)

\[ \frac{dv}{dx} = -\frac{1}{3} \frac{dy}{dx} \frac{dy}{dx} \]

\[ -\frac{1}{3} \frac{dy}{dx} + \frac{6}{x} \left( -\frac{1}{3} y^{-\frac{1}{3}} \right) = 3 \left( -\frac{1}{3} \right) y^{-\frac{1}{3}} y^{\frac{1}{3}} \]

\[ \frac{dv}{dx} - \frac{2}{x} v = -1 \]

This is linear. An integrating factor is \( e^{\int \frac{2}{x} dx} = e^{-2 \ln x} = x^{-2} \)

\[ \frac{1}{x^2} \frac{dv}{dx} - \frac{2}{x^3} v = -\frac{1}{x^2} \]

\[ \frac{d}{dx} \left( \frac{1}{x^2} v \right) = -\frac{1}{x^2} \]

\[ \frac{1}{x^2} v = \int -\frac{1}{x^2} dx = \frac{1}{x} + C \]

\[ v = x + Cx^2 \]

\[ y^{-\frac{1}{3}} = x + Cx^2 \]

\[ y = \frac{1}{(x + Cx^2)^3} \]

Also, \( y = 0 \) is a solution.
6. You prefer to drink your coffee at 60°C and exactly 10 minutes after it is made. You observe that in a 30°C room your coffee cools from 60°C to 50°C in 20 minutes. If you are in a room at 30°C, what temperature should you make your coffee so that you can enjoy it exactly as you like it? Simplify your answer as much as possible.

\[ \frac{dT}{dt} = k(T - 30) \]
\[ \frac{dT}{T - 30} = kdt \]
\[ \log |T - 30| = kt + C \]
\[ T - 30 = Ce^{kt} \]
\[ T = Ce^{kt} + 30 \]

Given that when T(0) = 60
then T(20) = 50

\[ 60 = T(0) = C + 30 \]
\[ 30 = C \]
\[ T(t) = 30e^{kt} + 30 \]

\[ 50 = T(20) = 30e^{20k} + 30 \]

\[ 20 = e^{20k} \]
\[ \log_{\frac{20}{30}}^{20} = k \]

\[ T(t) = 30e^{k t} \]
\[ (60) = T(10) = 30e^{\frac{1}{2} \log_{3}^{20} t} + 30 \]

\[ 30 = 30e^{\frac{1}{2} \log_{3}^{20} t} = \sqrt{3} \]
\[ \frac{30}{\sqrt{3}} = C \]

\[ T(0) = \frac{30}{\sqrt{3}} + 30 = 30(\sqrt{3} + 1) \]

7. Use the Euler method with stepsize $h = .1$ to estimate $y(1.2)$ for the initial value problem

$y' = x + y \quad y(1) = 1.$

$y_0 = 1 \quad x_0 = 1 \quad x_1 = 1.1 \quad x_2 = 1.2$

$y_1 = y_0 + h \cdot f(x_0, y_0) \quad \text{here } f(x, y) = x + y$

$y_1 = 1 + 0.1(1 + 1) = 1.2$

$y_2 = y_1 + h \cdot f(x_1, y_1) = 1.2 + 0.1(1.1 + 1.2)$

$= 1.2 + 0.1(2.3)$

$= 1.2 + 0.23$

$= 1.43$
8. Solve.

\[(x^2 + 1) \frac{dy}{dx} + 3xy = 6x\]

\[\frac{dy}{dx} + \frac{3x}{x^2 + 1} y = \frac{6x}{x^2 + 1}\]

This is linear. Integrating factor is

\[e^{\int \frac{3x}{x^2 + 1} \, dx} = e^{\frac{3}{2} \int \frac{2x}{x^2 + 1} \, dx} = e^{\frac{3}{2} \ln(x^2 + 1)}\]

\[= (e^{\ln(x^2 + 1)})^{\frac{3}{2}} = (x^2 + 1)^{\frac{3}{2}}\]

\[(x^2 + 1)^{\frac{3}{2}} \frac{dy}{dx} + 3x(x^2 + 1)^{\frac{1}{2}} y = 6x(x^2 + 1)^{\frac{1}{2}}\]

\[\frac{d}{dx} \left( (x^2 + 1)^{\frac{3}{2}} y \right) = 6x(x^2 + 1)^{\frac{1}{2}}\]

\[(x^2 + 1)^{\frac{3}{2}} y = \int 6x(x^2 + 1)^{\frac{1}{2}} \, dx\]

\[(x^2 + 1)^{\frac{3}{2}} y = 2(x^2 + 1)^{\frac{3}{2}} + C\]

\[y = 2 + \frac{C}{(x^2 + 1)^{\frac{3}{2}}}\]
9. Suppose $w(x)$ solves the initial value problem
\[
\frac{dw}{dx} + p(x)w = q(x)
\]
\[w(0) = 0\]

while $z(x)$ solves the initial value problem
\[
\frac{dz}{dx} + p(x)z = 0
\]
\[z(0) = y_0.\]

Show that $y(x) = w(x) + z(x)$ solves the initial value problem
\[
\frac{dy}{dx} + p(x)y = q(x)
\]
\[y(0) = y_0.\]

**First check to see if $y$ satisfies the eq'n:**
\[
\frac{dy}{dx} + p(x)y = \frac{dw}{dx}(w + z) + p(w + z) = \frac{dw}{dx} + \frac{dz}{dx} + pw + pz
\]
\[= \frac{dw}{dx} + pw + \frac{dz}{dx} + pz
\]
\[= q(x) + 0 = q(x).
\]

**Check that $y$ satisfies the initial condition:**
\[y(0) = w(0) + z(0) = 0 + y_0 = y_0.\]
10. Match the slope fields to the equations. (2 pts. each)

(a) \( y' = y \)
(b) \( y' = y^2 \)
(c) \( y' = 0.2x \)
(d) \( y' = x \)
(e) \( y' = 1 \)

This is the slope field of equation (a).
This is the slope field of equation (c).
This is the slope field of equation (d).
This is the slope field of equation (b).
This is the slope field of equation (e).