Differential Equations - MATH 240  
Final Exam, May 15, 2013

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Show all your work in the space provided under each question. Each problem is worth 10 points.

1. Find the general solution: $2x + 1 + 2y \frac{dy}{dx} = 0$.
   - This is separable. (It's also exact.)

   \[
   2y \frac{dy}{dx} = -(2x+1) \\
   \int 2y \, dy = - \int (2x+1) \, dx \\
   y^2 = -x^2 - x + C
   \]
2. Solve.

\[ x \frac{dy}{dx} = y + \sqrt{x^2 - y^2} \quad y(1) = 0. \]

\[ \frac{dy}{dx} = \frac{y + \sqrt{x^2 - y^2}}{x} \quad \text{this is homogeneous} \]

Substitute \( y = xv \), \( \frac{dy}{dx} = v + xv' \) and get

\[ v + xv' = x \frac{dv}{dx} + \frac{x^2 - xv^2}{x} \]

\[ v + xv' = v + \sqrt{1 - v^2} \quad \text{(note } x > 0 \text{ since initial condition is at } x = 1) \]

\[ xv' = \sqrt{1 - v^2} \]

\[ \int \frac{dv}{\sqrt{1 - v^2}} = \int \frac{dx}{x} \]

\[ \arcsin v = \ln|x| + c \]

Since \( v = \frac{y}{x} \) this is \( \arcsin \frac{y}{x} = \ln|x| + c \)

Plug in \( x = 1, y = 0 \) \( \arcsin 0 = \ln 1 + c \)

\[ 0 = 0 + c \]

\[ c = 0 \]

So \( \arcsin \frac{y}{x} = \ln x \)

\[ \frac{y}{x} = \sin(\ln x) \]

\[ y = x \sin(\ln x) \]
3. Solve.

\[ x'' - 16x' + 63x = \delta(t-4) \quad x(0) = 0, \quad x'(0) = 0. \]

\[ s^2X - 16sX + 63X = e^{-4s} \]

\[ (s^2 - 16s + 63)X = e^{-4s} \]

\[ X = \frac{e^{-4s}}{s^2 - 16s + 63} \]

\[ X = \frac{e^{-4s}}{(s-7)(s-9)} \]

\[ \frac{1}{(s-7)(s-9)} = \frac{A}{s-7} + \frac{B}{s-9} \]

\[ 1 = A(s-9) + B(s-7) \]

\[ s-9 \quad 1 = B(s-9) \quad s-7 \quad 1 = A(s-7) \]

\[ \frac{1}{2} = B \quad -\frac{1}{2} = A \]

\[ \frac{1}{(s-7)(s-9)} = -\frac{1}{2} \frac{1}{s-7} + \frac{1}{2} \frac{1}{s-9} \]

Thus,

\[ X = -\frac{1}{2} e^{-4s} + \frac{1}{2} e^{-4s} \]

\[ X(t) = -\frac{1}{2} u(t-4) e^{7(t-4)} + \frac{1}{2} u(t-4) e^{9(t-4)} \]
4. Find the general solution.

\[ x \frac{dy}{dx} + 3y = 3xy^{-\frac{1}{3}} \]

Bernoulli eq'n, \( v = y^{1-\left(\frac{1}{3}\right)} = y^{\frac{2}{3}} \)

\[ \frac{dv}{dx} = \frac{4}{3} y^{\frac{2}{3}} \frac{dy}{dx} \]

\[ \frac{4}{3} y^{\frac{2}{3}} \frac{dx}{dx} + \frac{3}{x} \cdot \frac{4}{3} y^{\frac{2}{3}} y = 3 \cdot \frac{4}{3} y^{\frac{2}{3}} y^{\frac{1}{3}} \]

\[ \frac{dv}{dx} + \frac{4}{3} v = 4 \]

This is linear. Integrating factor is \( \int e^{\frac{4}{3}dx} = (e^{\frac{4}{3}x}) \)

\[ = e^{x^4} \]

\[ x^4 \frac{dv}{dx} + 4x^3 v = 4x^4 \]

\[ \int \frac{dv}{dx} (v x^4) = \int 4x^4 \, dx \]

\[ v x^4 = \frac{4}{5} x^5 + C \]

\[ v = \frac{4}{5} x + \frac{C}{x^4} \]

\[ y = v^{\frac{3}{4}} = \left( \frac{4}{5} x + \frac{C}{x^4} \right)^{\frac{3}{4}} \]
5. Find the general solution. Assume $x > 0$.

$$x^2y'' + 4xy' + 2y = \log x$$

$$y - y' + y_0$$

$$(x^2 - 4r + 2) = 0$$

Cauchy-Euler eqn.

Indicial eqn is $r(r-1) + 4r + 2 = 0$

$$r^2 + 3r + 2 = 0$$

$$(r + 1)(r + 2) = 0$$

$$r = -1, -2$$

$$y_h = c_1 \frac{1}{x} + c_2 \frac{1}{x^2}$$

Rewrite

$$y'' + \frac{4}{x}y' + \frac{2}{x^2}y = \frac{1}{x} \log x$$

to find $y_p$ use variation of parameters

$$W(\frac{1}{x}, \frac{1}{x^2}) = \det \begin{bmatrix} \frac{1}{x} & \frac{1}{x^2} \\ -\frac{1}{x^2} & \frac{1}{x^3} \end{bmatrix} = \frac{-2}{x^4} - (-\frac{1}{x^4}) = \frac{1}{x^4}$$

$$y_p = \frac{1}{x} \int \frac{-\frac{x^2}{x^2} \log x}{-\frac{1}{x^4}} \, dx + \frac{1}{x^2} \int \frac{\frac{1}{x^2} \log x}{-\frac{1}{x^4}} \, dx$$

$$= \frac{1}{x} \left[ \log x \, dx \right] - \frac{1}{x^2} \left[ x \log x - x \right]$$

$$= \log x - 1 - \frac{1}{x^2} \left[ \frac{x^2}{2} \log x - \frac{x^2}{4} \right]$$

$$= \log x - \frac{1}{2} \log x - \frac{3}{4} = \frac{1}{2} \log x - \frac{3}{4}$$

So

$$y = \frac{c_1}{x} + \frac{c_2}{x^2} + \frac{1}{2} \log x - \frac{3}{4}$$
6. Use power series to solve the differential equation.

\[ y'' + xy' - 2y = 0 \quad y(0) = 1, \quad y'(0) = 0 \]

\[ y = \sum_{n=0}^{\infty} a_n x^n \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad y'' = \sum_{n=2}^{\infty} (n-1) n a_n x^{n-2} \]

Plug in:

\[ \sum_{n=2}^{\infty} (n-1) n a_n x^{n-2} + x \sum_{n=1}^{\infty} n a_n x^{n-1} - 2 \sum_{n=0}^{\infty} a_n x^n = 0 \]

\[ \sum_{n=2}^{\infty} (n-1) n a_n x^{n-2} + \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} 2a_n x^n = 0 \]

\[ m = n-2 \quad m+1 = n-1 \quad m = n \]

\[ \sum_{m=0}^{\infty} (m+1)(m+2) a_{m+2} x^m + \sum_{m=1}^{\infty} m a_m x^m - \sum_{m=0}^{\infty} 2a_m x^m = 0 \]

\[ 2a_2 - 2a_0 + \sum_{m=2}^{\infty} (m+1)(m+2) a_{m+2} + ma_m - 2a_m x^m = 0 \]

So \( 2a_2 - 2a_0 = 0 \) and \( a_{m+2} = \frac{(m-2)}{(m+1)(m+2)} a_m \)

i.e. \( a_2 = a_0 \)

\[ a_1 = 0 \]

\[ a_3 = \frac{-6}{3 \cdot 4} a_2 = 0 \]

\[ a_4 = \frac{-6}{3 \cdot 4} a_2 = 0 \]

\[ a_5 = 0, \quad a_6 = 0, \quad \text{etc.} \]

So \( y = 1 + x^2 \)
7. Find the general solution.

\[ y'' - y = e^x \]

\[(D^2 - 1)y = 0 \]

\[(D - 1)(D + 1)y = 0 \]

\[ D = 1 \quad D = -1 \pm \frac{\sqrt{13}}{2} = -\frac{1}{2} \pm \frac{\sqrt{13}}{2}i \]

so \( y_h = c_1 e^x + c_2 e^{-\frac{1}{2}x} \cos \frac{\sqrt{13}}{2}x + c_3 e^{-\frac{1}{2}x} \sin \frac{\sqrt{13}}{2}x \)

for \( y_p \) try \( y_p = Axe^x \)

Then \( y_p' = Ae^x + Axe^x \)

\( y_p'' = Ae^x + Ae^x + Axe^x = 2Ae^x + Axe^x \)

\( y_p''' = 2Ae^x + Ae^x + Axe^x = 3Ae^x + Axe^x \)

plug in: \( y_p''' = 3Ae^x + Axe^x - Axe^x = e^x \)

\( 3A = 1 \quad A = \frac{1}{3} \)

so \( y_p = \frac{1}{3}xe^x \)

Then \( y = y_h + y_p = c_1 e^x + c_2 e^{-\frac{1}{2}x} \cos \frac{\sqrt{13}}{2}x + c_3 e^{-\frac{1}{2}x} \sin \frac{\sqrt{13}}{2}x + \frac{1}{3}xe^x \)
8. Consider the initial value problem

\[ x'' + 2x' + 2x = h(t), \quad x(0) = 1, \quad x'(0) = 0, \]

where \( h(t) \) is an unknown function. Solve this, expressing a part of your answer as a convolution integral involving the function \( h(t) \).

\[
\begin{align*}
S^2 x + 2 & = S^2 x - 3 \cdot h - 0 + 2(5x - 1) + 2 x - 8 \delta(h) \\
(5^2 + 2s + 2) x & = 5 + 2 + 8 \delta(h) \\
x & = \frac{s + 2}{s^2 + 2s + 2} + \frac{1}{s^2 + 2s + 2} 8 \delta(h) \\
x & = \frac{s + 2}{(s+1)^2 + 1^2} + \frac{1}{(s+1)^2 + 1^2} 8 \delta(h) \\
x(t) & = e^{t \cos t} + e^{-t \sin t} + e^{t \sin t} \ast h \\
x(t) & = e^{t \cos t} + e^{-t \sin t} + \int_0^t e^{-(t-t') \sin(t-t')} h(t') \, dt'
\end{align*}
\]
9. The system

\[
\begin{align*}
\frac{dx}{dt} &= xy - 4 \\
\frac{dy}{dt} &= x - y
\end{align*}
\]

has an equilibrium point in the third quadrant. Find and classify this equilibrium point.

\[xy - 4 = 0 \quad \text{so} \quad x = y \quad \text{then} \quad x^2 - 4 = 0 \quad x = \pm 2 \quad \text{so pt. in 3rd quad.,}
\]

\[f(x, y) = xy - 4 \quad \frac{df}{dx} = x \quad \frac{df}{dx}(-2, 2) = 2 \quad \frac{df}{dy} = y \quad \frac{df}{dy}(-2, 2) = -2
\]

\[g(x, y) = x - y \quad \frac{dg}{dx} = 1 \quad \frac{dg}{dx}(-2, 2) = 1 \quad \frac{dg}{dy} = -1 \quad \frac{dg}{dy}(-2, 2) = -1
\]

\[\text{so} \quad x - 4 = -2(x + 2) - 2(y + 2) + \cdots \\
\text{and} \quad x - y = 1(x + 2) - 1(y + 2) + \cdots
\]

\[\text{so near } (-2, -2) \text{ system behaves like}
\]

\[
\frac{dx}{dt} = -2x - 2y \quad \text{near } (0, 0)
\]
\[
\frac{dy}{dt} = x - y
\]

\[T = -3 \quad D = 4
\]
\[T^2 = (-3)^2 = \frac{9}{4} < 4 = D
\]

inward spiral
10. Find all solutions to the boundary value problem:

\[ y'' + 4y = \sin(x) \quad y(0) = 0, \quad y(\pi) = 0. \]

\[ D^2 + 4D \equiv 0 \]

\[ D = \pm 2i \]

\[ y_h = c_1 \cos 2x + c_2 \sin 2x \]

\[ y_p = A \cos x + B \sin x \]

\[ y_p' = -A \sin x + B \cos x \]

\[ y_p'' = -A \cos x - B \sin x \]

\[-A \cos x - B \sin x + 4(A \cos x + B \sin x) = \sin x\]

\[ 3A \cos x = 0 \cos x \]

so \( A = 0 \)

\[ 3B \sin x = \sin x \]

so \( B = \frac{1}{3} \)

Therefore \( y = y_h + y_p = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{3} \sin x \)

\[ 0 = y(0) = c_1 + 0 \cdot c_2 \Rightarrow 0 = c_1 \]

so \( y = c_2 \sin 2x + \frac{1}{3} \sin x \)

\[ 0 = y(\pi) = c_2 \cdot 0 + 0 \]

anything is \( c_2 \)

so \( y = c_2 \sin 2x + \frac{1}{3} \sin x \)
11. Find the Taylor series of the function \( \sinh(x) \) about \( x_0 = 0 \). You may use your knowledge of the Taylor series of \( e^x \), \( \sin(x) \), \( \cos(x) \), and \( \frac{1}{1-x} \). What is the radius of convergence of the series?

\[
\sinh x = \frac{e^x - e^{-x}}{2}
\]

\[
= \frac{1}{2} \left( 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \right) - \frac{1}{2} \left( 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots \right)
\]

\[
= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \ldots
\]

Since both \( e^x \) and \( e^{-x} \) have series with infinite radius of convergence, the power series of \( \sinh x \) also has an infinite radius of convergence.
12. Find the roots of the indicial equation about $x_0 = 0$ for

$$x^2y'' + (2x^2 + 6x)y' + (2x^2 + 3x + 4)y = 0.$$ 

The indicial equation is:

$$\alpha(\alpha - 1) + 6\alpha + 4 = 0$$

$$\alpha^2 + 5\alpha + 4 = 0$$

$$(\alpha + 1)(\alpha + 4) = 0$$

$$\alpha = -1, -4$$
13. Match the slope fields to the equations.

A. \( y' = y \)  
B. \( y' = -1 \)  
C. \( y' = xy \)  
D. \( y' = x + y \)  
E. \( y' = \cos(x) \)

This is the slope field of equation C.  
This is the slope field of equation B.  
This is the slope field of equation E.  
This is the slope field of equation D.  
This is the slope field of equation A.
14. Match the equations to the poles of the Laplace transform of the solution.

A. \( x'' + 10x' + 34x = 0 \) \( x(0) = 10, \ x'(0) = 0 \)
B. \( x'' + 3x' + 2x = 0 \) \( x(0) = 10, \ x'(0) = 0 \)
C. \( x'' + 2x' + 2x = \cos t \) \( x(0) = 10, \ x'(0) = 0 \)
D. \( x'' + 4x' + 4x = 0 \) \( x(0) = 10, \ x'(0) = 0 \)
E. \( x'' - x' + 10x = 0 \) \( x(0) = 10, \ x'(0) = 0 \)

These are the poles of equation C. These are the poles of equation B.

These are the poles of equation E. These are the poles of equation A.

These are the poles of equation D.
15. Match the graphs to the equations. Each graph ranges from 0 to $6\pi$ on the horizontal scale and from $-10$ to 10 on the vertical scale.

A. $x'' + x = 0 \quad x(0) = 9, \ x'(0) = 0$
B. $x'' + x = 0 \quad x(0) = 0, \ x'(0) = 10$
C. $x'' + x = \cos(t) \quad x(0) = 0, \ x'(0) = 0$
D. $x'' + 36x = 20\cos(6.5t) \quad x(0) = 0, \ x'(0) = 0$
E. $x'' + .2x + x = 0 \quad x(0) = 0, \ x'(0) = 10$

This is the graph of equation D.

This is the graph of equation B.

This is the graph of equation A.

This is the graph of equation E.

This is the graph of equation C.