Differential Equations - MATH 240
Exam 2
April 2, 2002

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Show all your work in the space provided under each question. Each problem is worth 12 points except for problem 8 which is worth 16 points.

1. Find the general solution of \( y'' - 4y' + 5y - 2y = 0 \).

   Characteristic eqn is \( D^3 - 4D^2 + 5D - 2 = 0 \)

   Note that \( D = 1 \) is a root. \( 1^3 - 4 \cdot 1^2 + 5 \cdot 1 - 2 = 0 \)

   So \( D - 1 \) divides \( D^3 - 4D^2 + 5D - 2 \).

   \[
   \begin{align*}
   D^2 - 3D + 2 & \quad \text{divides} \quad D^3 - 4D^2 + 5D - 2 \\
   D - 1 & \quad \text{divides} \quad D^3 - 4D^2 + 5D - 2 \\
   \end{align*}
   \]

   So \( (D - 1)(D^2 - 3D + 2) = 0 \)

   \( (D - 1)(D - 1)(D - 2) = 0 \)

   \( D = 1 \) repeated

   \( D = 2 \)

   \[
   y = C_1 e^x + C_2 xe^x + C_3 e^{2x}
   \]
2. Find the general solution to

\[ y'' + 3y' - 4y = e^x. \]

\[
\begin{align*}
    p^2 + 3p - 4 &= 0 \\
    (p + 4)(p - 1) &= 0 \\
    p &= -4, 1
\end{align*}
\]

so \( y_h = c_1 e^{-4x} + c_2 e^x \)

to find \( y_p \) try \( y_p = Axe^x \)

then \( y_p' = Ae^x + Axe^x \)
\[
y_p'' = Ae^x + Ae^x + Axe^x = 2Ae^x + Axe^x
\]

so \( 2Ae^x + Axe^x + 3(Ae^x + Axe^x) - 4(Axe^x) = e^x \)
\[
2Ae^x + Axe^x + 3Ae^x + 3Axe^x - 4Axe^x = e^x
\]

\( 5Ae^x = e^x \)

\[ A = \frac{1}{5} \]

so \( y_p = \frac{1}{5}xe^x \)

so \( y = y_h + y_p = c_1 e^{-4x} + c_2 e^x + \frac{1}{5}xe^x \)
3. Consider the second order linear constant coefficient homogeneous equation

\[ mx'' + cx' + kx = 0 \]

(4 pts) (a) Assuming \( m > 0, c \geq 0, k > 0 \), what conditions on \( m, c, \) and \( k \) result in solutions which go to 0 without oscillating? Explain.

\[ D = \frac{-c \pm \sqrt{c^2 - 4km}}{2m} \]

If the eqn has 2 distinct real roots, that is, if \( c^2 - 4km > 0 \), these are both negative and the solution is of the form \( x = c_1 e^{r_1 t} + c_2 e^{r_2 t} \) where \( r_1, r_2 \) are the roots. If \( c^2 - 4km = 0 \), \( x = c_1 e^{r_1 t} + c_2 e^{r_2 t} \) where \( r_1 \) is the root. In either case solutions \( \to 0 \).

(4 pts) (b) Again, assuming \( m > 0, c \geq 0, k > 0 \), what conditions on \( m, c, \) and \( k \) result in solutions which go to 0 while oscillating? Explain.

\[ D = \frac{-c \pm \sqrt{c^2 - 4km}}{2m} \quad \text{if} \quad c > 0, \quad c^2 - 4km < 0 \]

then \( D \) has the form \( D = p \pm iq \) where \( p < 0 \).

then \( x(t) = c_1 e^{pt} \cos qt + c_2 e^{pt} \sin qt \)

which oscillates but decays to 0.

(4 pts) (c) Again, assuming \( m > 0, c \geq 0, k > 0 \), what conditions on \( m, c, \) and \( k \) result in solutions which oscillate without decaying? Explain.

This is similar to (b), but here \( c = 0 \)

so that \( D = \pm \sqrt{-4km} = \pm iq \)

so if \( c = 0 \) then \( x(t) = c_1 e^{pt} + c_2 e^{pt} \)

which oscillates without decaying.
4. A mass of 64 kg is attached to a spring which causes it to stretch 19.6 m. After the system comes to equilibrium, it is set into motion with an initial displacement of 10 m and an initial velocity. What should the initial velocity be so that the amplitude of the resulting motion is 12 m? Take \( g = 9.8 \, \text{m/s}^2 \).

\[
mg = k \Delta l
\]

\[
64(9.8) = k(19.6)
\]

\[
64 = 2k
\]

\[
32 = k
\]

\[
64x'' + 32x = 0 \quad x(0)=10 \quad x'(0)=?
\]

Use energy.

Initially \( E = \frac{1}{2}(64) \, x'(0)^2 + \frac{1}{2} \, 32 \, (10)^2 \)

At max amp \( E = \frac{1}{2} \, 32 \, (12)^2 \)

Since energy is conserved

\[
\frac{1}{2} \, (64) \, x'(0)^2 + \frac{1}{2} \, 32 \, (10)^2 = \frac{1}{2} \, 32 \, (12)^2
\]

\[
64 \, x'(0)^2 = 32 \, (144 - 100)
\]

\[
x'(0)^2 = \frac{1}{2} \cdot 44
\]

\[
x'(0)^2 = 22
\]

\[
x'(0) = \pm \sqrt{22} \, \text{m/sec}
\]
5. Find the general solution to the system

\[
\frac{dx}{dt} = 4x + y \\
\frac{dy}{dt} = 3x + 2y.
\]

**Solve first equation for y:**

\[
y = \frac{dx}{dt} - 4x
\]

**Plug into 2nd:**

\[
\frac{d}{dt} \left( \frac{dx}{dt} - 4x \right) = 3x + 2 \left( \frac{dx}{dt} - 4x \right)
\]

\[
\frac{d^2x}{dt^2} - 4 \frac{dx}{dt} = 3x + 2 \frac{dx}{dt} - 8x
\]

\[
\frac{d^2x}{dt^2} - 6 \frac{dx}{dt} + 5x = 0
\]

\[
D^2 - 6D + 5 = 0
\]

\[
(b-1)(b-5) = 0 \quad b = 1, 5
\]

So

\[
x(t) = c_1e^{1t} + c_2e^{5t}
\]

Then

\[
x(t) = \frac{dx}{dt} - 4x = c_1e^{1t} + 5c_2e^{5t} - 4(c_1e^{1t} + c_2e^{5t})
\]

\[
= -3c_1e^{1t} + c_2e^{5t}
\]
another way.

5. Find the general solution to the system

\[
\begin{align*}
\frac{dx}{dt} &= 4x + y \\
\frac{dy}{dt} &= 3x + 2y.
\end{align*}
\]

Solve second eq'n for \(x\): \(\frac{1}{3}(\frac{dy}{dt} - 2y) = x\)

Plug into first eq'n:

\[
\frac{d}{dt} \left( \frac{1}{3}(\frac{dy}{dt} - 2y) \right) = 4 \left[ \frac{1}{3}(\frac{dy}{dt} - 2y) \right] + y
\]

\[
\frac{1}{3} \frac{d^2 y}{dt^2} - \frac{2}{3} \frac{dy}{dt} = \frac{4}{3} \frac{dy}{dt} - \frac{8}{3} y + y
\]

\[
\frac{1}{3} \frac{d^2 y}{dt^2} - 2 \frac{dy}{dt} + \frac{8}{3} y = 0
\]

\[
\frac{1}{3} b^2 - 2b + \frac{8}{3} = 0
\]

\[
b^2 - 6b + 8 = 0 \quad (b - 1)(b - 8) = 0 \quad b = 1, 8
\]

\[y = c_1 e^x + c_2 e^{6x}\]

\[x = \frac{1}{3} \left( \frac{dy}{dt} - 2y \right) = \frac{1}{3} \left( c_1 e^x + 5c_2 e^{6x} \right) - \frac{2}{3} \left( c_1 e^x + c_2 e^{6x} \right)
\]

\[= -\frac{1}{3} c_1 e^x + c_2 e^{6x}\]
6. Solve the initial value problem

\[ y'' + 9y = 0 \quad y(\pi) = 0 \quad y'(\pi) = 1. \]

\[ D^2 + 9 = 0 \]
\[ D = \pm 3i \]

\[ y = c_1 \cos 3t + c_2 \sin 3t \]

\[ 0 = c_1 \cos 3\pi + c_2 \sin 3\pi \]
\[ 0 = c_1 (-1) + c_2 (0) \]
so \( c_1 = 0 \)

so \( y = c_2 \sin 3t \)

\[ y'' = 3c_2 \cos 3t \]

\[ 1 = y'(\pi) = 3c_2 \cos 3\pi = 3c_2 (-1) = -3c_2 \]
\[ -\frac{1}{3} = c_2 \]

so \( y = -\frac{1}{3} \sin 3t \)
7. Find the general solution of

\[ y'' - 2y' + y = \frac{e^x}{1 + x^2}. \]

\[ D^2 - 2D + 1 = 0 \]
\[ (D-1)^2 = 0 \quad D = 1 \text{ repeated} \]

\[ y_h = c_1 e^x + c_2 x e^x \]

to find \( y_p \) use variation of parameters

\[ W(e^x, xe^x) = \text{det} \begin{vmatrix} e^x & xe^x \\ e^x & e^x + xe^x \end{vmatrix} = e^x + xe^x - xe^x = e^x \]

\[ y_p = y_1 \int \frac{-y_2 g}{W(y_1, y_2)} \, dx + y_2 \int \frac{y_1 g}{W(y_1, y_2)} \, dx \]

\[ = e^x \int \frac{-xe^x}{e^x} \frac{e^x}{1 + x^2} \, dx + xe^x \int \frac{e^x}{e^x} \frac{e^x}{1 + x^2} \, dx \]

\[ = -e^x \int \frac{x}{1 + x^2} \, dx + xe^x \int \frac{1}{1 + x^2} \, dx \]

\[ = -\frac{e^x}{2} \log(1 + x^2) + xe^x \arctan x \]

\[ \therefore y = y_h + y_p \]

\[ = c_1 e^x + c_2 x e^x - \frac{e^x}{2} \log(1 + x^2) + xe^x \arctan x \]
8. Match the graphs to the equations. On each of the graphs the horizontal axis runs from 0 to 20. Each correct answer is worth 4 points.

1. $x'' + x = \cos t, \ x(0) = x'(0) = 0$  
   Its graph is [Image C]

2. $x'' + x = 0, \ x(0) = 0, \ x'(0) = 9$  
   Its graph is [Image A]

3. $5x'' + x' + 10x = 0, \ x(0) = 0, \ x'(0) = 10$  
   Its graph is [Image D]

4. $x'' + 400x = 200 \cos 18t, \ x(0) = x'(0) = 0$  
   Its graph is [Image B]