Show all your work in the space provided under each question. Each problem is worth 12 points, except for problem 8 which is worth 16 points.

1. Find the general solution to the differential equation

\[(x^2 + 1) \frac{dy}{dx} + 3xy = 6x\]

\[\frac{x^2 + 1}{x - 3y} = (x^2 + 3y)\text{ separable}\]
\[
\int \frac{dy}{x - 3y} = \int \frac{dx}{x^2 + 1}
\]
\[-\frac{1}{3} \log |6 - 3y| = \frac{1}{2} \log (x^2 + 1) + C
\]
\[\log |6 - 3y| = -\frac{3}{2} \log (x^2 + 1) + C
\]
\[6 - 3y = e^{C (x^2 + 1)^{-3/2}}
\]
\[6 - 3y = C(x^2 + 1)^{3/2}
\]
\[y = 2 + \frac{k}{(x^2 + 1)^{3/2}}, \quad k \neq 0
\]
2. Find the general solution to the differential equation

\[(2x + y^2) + 2xy \frac{dy}{dx} = 0\]

\[\frac{\partial}{\partial y}(2x + y^2) = 2y \quad \frac{\partial}{\partial x}(2xy) = 2y\]

So, it is exact.

Solve it is \( F(x, y) = C \) where

\[\frac{\partial F}{\partial x} = 2x + y^2 \quad \frac{\partial F}{\partial y} = 2x - y\]

\[\frac{\partial F}{\partial x} = 2x + 1 \implies F(x, y) = x^2 + x + y^2 + k(y)\]

then \( 2x + y = \frac{\partial F}{\partial y} = 0 - 2xy + k'(y) \)

Therefore \( k'(y) = 0 \)

So \( k(y) = C \)

can take \( C = 0 \) so \( F(x, y) = x^2 + xy^2 \)

\[\text{set } k = x^2 + xy^2 = C\]
3. The spread of certain diseases in a population can be described by the equation \( \frac{dP}{dt} = \alpha P(1 - P) \) where \( P \) is the proportion of the infected individuals in the population and \( \alpha \) is a positive constant which depends on the disease. (Thus, \( \frac{dP}{dt} \) represents the rate of spread of the disease.) If one member of the population becomes infected, what is the fate of the population? To receive credit you must explain using graphs and mathematical analysis.

Critical pts are \( P = 0, P = 1 \).

If \( P < 0 \), \( \frac{dP}{dt} = \alpha (P)(1 - P) = \alpha P - \alpha P^2 < 0 \),

\( P \) is dec.

If \( 0 < P < 1 \), \( \frac{dP}{dt} = \alpha P (1 - P) - \alpha (1)(1) > 0 \)

\( P \) inc.

If \( P > 1 \), \( \frac{dP}{dt} = \alpha P (1 - P) = \alpha (P - 1)(P - 1) < 0 \) \( P \) dec.

Note that \( P > 1 \) or \( P < 0 \) doesn't really make sense for this problem. However, if one individual becomes infected, \( P(0) > 0 \) (and of course \( P(0) \leq 1 \)) we have \( P(1) = 1 \); that is, all eventually are infected.
4. Solve \( x^2 y' = y^3 - 2xy \).

\[
\begin{align*}
\frac{d}{dx} \left( x^2 y' \right) &= y^3 - 2xy \\
y' + \frac{2}{x} y &= \frac{1}{x^2} y^3 \\
\text{Exponential} &
\end{align*}
\]

\[
\frac{1}{2} v^{-\frac{1}{2}} \frac{dv}{dx} + \frac{1}{x} v^{-\frac{1}{2}} - \frac{1}{x^2} v^{-\frac{1}{2}}
\]

\[-\frac{1}{2} v^{-\frac{1}{2}} \left( \frac{dv}{dx} + \frac{1}{x} v^{-\frac{1}{2}} \right) = \frac{1}{x^2}
\]

\[
\frac{dv}{dx} = -\frac{1}{2} v^{-\frac{1}{2}} \frac{dv}{dx}
\]

Indefinite factor is \( e^{-\frac{1}{2}x} \).

\[
\frac{d}{dx} \left( \frac{1}{x^2} v \right) = -\frac{1}{x^2}
\]

\[
\frac{1}{x^2} v = \frac{1}{3} x^{-\frac{3}{2}} + C
\]

\[
v = \frac{1}{3} x^{-\frac{3}{2}} + C x^4
\]

\[
y = \left( \frac{1}{3 x^2 + C x^4} \right)^{\frac{1}{2}}
\]

\[\text{Note: we divided by 0. Check } y = 0. \text{ It's a soln, also}
\]

\[\therefore y = \left( \frac{1}{3 x^2 + C x^4} \right)^{\frac{1}{2}} \text{ or } y = 0\]
5. Use the Euler method with stepsize $h = .5$ to estimate $y(1)$ for the initial value problem $y' = x^2 + y \quad y(0) = 1$.

\[ y_1 = y_0 + h f(x_0, y_0) \]
\[ = 1 + .5 (0^2 + 1) - 1.5 \]

\[ y_2 = 1.5 + .5 (0.5^2 + 1.5) \]
\[ = 1.5 + .5 (0.25 + 1.5) \]
\[ = 1.5 + .5 (1.75) \]
\[ = 1.5 + .875 \]
\[ = 2.375 \]

$y(1) \approx 2.375$
6. (a) Simplify. Express your answer in the form \( a + bi \).

\[
(2 + 4i)e^{\frac{2\pi i}{3}}
\]

\[
(2 + 4i) e^{\frac{2\pi i}{3}} = (2 + 4i) (\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}) = (2 + 4i) \left( -\frac{1}{2} + \frac{\sqrt{3}}{2} i \right)
\]

\[
= -1 - 2i + \sqrt{3} i - 2\sqrt{3}
\]

\[
= -1 - 2\sqrt{3} + (\sqrt{3} - 2)i
\]

(b) Write in the form \( A \cos(\omega x + \phi) \): \( 2\sqrt{3} \cos(2x) + 2 \sin(2x) \).

\[
2\sqrt{3} \cos 2x = \text{Re} \left( 2\sqrt{3} e^{2ix} \right) \quad 2 \sin 2x = \text{Re} \left( -2ie^{2ix} \right)
\]

\[
2\sqrt{3} \cos 2x + 2 \sin 2x = \text{Re} \left( 2\sqrt{3} \cos 2x - 2i \sin 2x \right)
\]

\[
= \text{Re} \left( 4e^{-\frac{\pi i}{6}} e^{2ix} \right)
\]

\[
= \text{Re} \left( 4e^{-\frac{\pi i}{6}} \left( \cos \left( \frac{11\pi}{6} \right) - i \sin \left( \frac{11\pi}{6} \right) \right) \right)
\]

\[
= 4 \cos \left( 2x + \frac{\pi}{6} \right)
\]

\[
\gamma = \sqrt{2^2 + (2\sqrt{3})^2} = \sqrt{4 + 12} = 4
\]

\[
\theta = \frac{-\pi}{6}
\]

\[
\Rightarrow \quad 2\sqrt{3} - 2i - 4e^{\frac{\pi i}{6}}
\]
7. Solve the differential equation

\[ 2xy \frac{dy}{dx} - 4x^2 = 3y^2 \]

\[ \frac{dy}{dx} = \frac{3y^2 + 4x^2}{2xy} \quad \text{homogeneous} \]

\[ y = x \cdot v \quad \Rightarrow \quad \frac{dv}{dx} = v + x \frac{dv}{dx} \]

\[ v + x \frac{dv}{dx} = \frac{3(2v)^2 + 4x^2}{2x \cdot xv} = \frac{3v^2 + 4}{2v} \]

\[ x \frac{dv}{dx} = \frac{3v^2 + 4}{2v} - \frac{2v}{2v} \cdot v \]

\[ x \frac{dv}{dx} = \frac{v^2 + 4}{2v} \]

\[ \int \frac{2v}{v^2 + 4} \, dv = \int \frac{1}{x} \, dx \]

\[ \log(v^2 + 4) = \log x + C \]

\[ v^2 + 4 = k \times \]

\[ \left( \frac{y}{x} \right)^2 + 4 = k \times \]

\[ y^2 + 4x^2 = k \times \]

need to check for singular solutions.

divided by v. If v = 0, y = 0, not a solution.
7. Solve the differential equation

\[ 2xy \frac{dy}{dx} - 4x^2 = 3y^2 \]

\[ \frac{dy}{dx} - \frac{2x}{y} = \frac{3}{2x} \]

\[ \frac{dy}{dx} - \frac{3}{2x} y = +2xy^{-1} \quad \text{Bernoulli} \]

\[ v = y^{1-(-1)} = y^2 \quad \Rightarrow \quad y = v^{\frac{1}{2}} \]

\[ \frac{1}{2} v^{-\frac{1}{2}} \frac{dv}{dx} - \frac{3}{2x} v^{\frac{1}{2}} = 2x v^{-\frac{1}{2}} \]

\[ \frac{1}{2} \frac{dv}{dx} - \frac{3}{2x} v = 2x \]

\[ \frac{dv}{dx} - \frac{3}{x} v = 4x \quad \text{linear} \]

Integrating factor: \( e^{\int -\frac{3}{x} dx} = e^{-3\ln x} = x^{-3} \)

\[ \frac{dv}{dx} x^{-3} - \frac{3}{x} x^{-3} v = 4 x^{-2} \]

\[ \frac{d}{dx} \left( v \frac{1}{x^3} \right) = \frac{4}{x^2} \]

\[ v \frac{1}{x^3} = -\frac{4}{x} + C \]

\[ v = -4x^2 + Cx^3 \]

\[ y = \sqrt{C x^3 - 4x^2} \]
8. Match the slope fields to the equations.

(a) \( y' = xy \)
(b) \( y' = x^2 - y^2 \)
(c) \( y' = y \)
(d) \( y' = .2x \)

This is the slope field of equation \( \text{a} \).  This is the slope field of equation \( \text{a} \).

This is the slope field of equation \( \text{b} \).  This is the slope field of equation \( \text{c} \).