Extra Credit Assignment #1: Reducible Second-Order Equations
Due in recitation Thursday, March 4, 2004

You are encouraged to collaborate with your colleagues. For credit, however, your final write-up must be done individually. Show all your work and make your presentation comprehensible.

A second-order differential equation has the general form

$$F(x, y, y', y'') = 0,$$

where $F$ is a function relating $x$, $y$, $y'$, and $y''$. If the independent variable $x$ or the dependent variable $y$ is not present in the differential equation, then the order of the equation can be reduced using a substitution.

Dependent variable $y$ missing: The differential equation has the form

$$F(x, y', y'') = 0.$$  

We can substitute $v = y'$, so $v' = y''$. The differential equation becomes

$$F(x, v, v') = 0,$$

which is a first-order equation. If this can be solved for the unknown function $v$, then the unknown function $y$ is

$$y(x) = \int y'(x) \, dx = \int v(x) \, dx.$$  

The solution $y$ will have two arbitrary constants (as is typical for solutions to second-order equations).

Example. Solve $xy'' + 2y' = 6x$ ($y$ is not explicitly present in this equation). Substituting $v = y'$ leads to

$$xv' + 2v = 6x \Rightarrow v' + \frac{2}{x}v = 6.$$  

This is a first-order linear equation; it’s solutions are

$$v = 2x + \frac{C_1}{x^2}.$$  

Now $v = y'$, so

$$y(x) = \int y'(x) \, dx = \int \left(2x + \frac{C_1}{x}\right) \, dx = x^2 + \frac{C_1}{x} + C_2,$$

where $C_1$ and $C_2$ are arbitrary constants.
**Independent variable $x$ missing:** The differential equation has the form

$$F(y, y', y'') = 0.$$  

For these types of equations, it’s useful to think of $x$ as a function of $y$, and thus $y'(x)$ can also be thought of as a function of $y$ (since $x$ is). We substitute

$$v = y' = \frac{dy}{dx} \Rightarrow y'' = \frac{dv}{dx} = \frac{dv}{dy} \frac{dy}{dx} = v \frac{dv}{dy};$$

we used the chain rule, since we are thinking of $y'$ as a function of $y$. The differential equation becomes

$$F(y, v, v\frac{dv}{dy}) = 0,$$

where $p$ is a function of $y$. If this can be solved for $v$, then we can find $x$ as a function $y$ by

$$x(y) = \int \frac{dx}{dy} dy = \int \left( \frac{dy}{dx} \right)^{-1} dx = \int \frac{1}{v(y)} dy.$$

This yields an implicit formula for $y$ which will include two arbitrary constants.

**Example.** Solve $yy'' = (y')^2$ ($x$ is not present in this equation). Substituting $v = y'$ leads to

$$yv\frac{dv}{dy} = v^2.$$

This is a first-order separable equation; its solutions are

$$v = C_1 y.$$  

Now $\frac{1}{v} = x'$, so

$$x(y) = \int x'(y) dy = \int \frac{1}{C_1 y} dy = \frac{1}{C_1} \ln |y| + C_2.$$  

We solve this solution for $y$ explicitly

$$\ln |y| = C_1 x - C_2 \Rightarrow y(x) = C_2 e^{C_1 x},$$

where $C_1$ and $C_2$ are arbitrary constants.
Solve the following problems.

1. Find all solutions to \( x^2 y'' + 3xy' = 2 \).

2. Find all solutions to \( y'' = (y')^2 \).

3. Suppose that a uniform flexible cable is suspended between two points \((-L, H)\) and \((L, H)\) at equal heights. Physical principles can be used to show that the shape of the hanging cable can be represented by the graph of a function \( y = y(x) \) satisfying the differential equation

\[
y'' = \sqrt{1 + (y')^2},
\]

where the constant \( a \) is determined by the properties of the cable. Find the shape function \( y \) for the hanging cable. (Note: the graph of the solution is called a catenary curve, and it does match very closely with a real hanging cable.)

4. In your calculus courses, you learned that the curvature \( \kappa \) of a curve \( y = y(x) \) at the point \((x, y)\) is given by

\[
\kappa = \frac{|y''(x)|}{\left[1 + (y'(x))^2\right]^{\frac{3}{2}}}.
\]

Assuming that \( y'' \geq 0 \), a curve with constant curvature \( r \) will satisfy

\[
\frac{1}{r} y'' = \left[1 + (y')^2\right]^{\frac{3}{2}}.
\]

Find the general solution to this differential equation. Give a geometric description of a curve with constant curvature \( r \)?