Written Assignment #3: Riccati Equations (Solutions)

1. Equations of the form \( \frac{dy}{dx} = A(x)y^2 + B(x)y + C(x) \) are called Riccati equations. If \( y_1(x) \) is a known particular solution to a Riccati equation, then the substitution \( v = y - y_1 \) will transform the Riccati equation into a Bernoulli equation.

(a) If \( v(x) = y(x) - y_1(x) \), then what do \( y(x) \) and \( y'(x) \) equal (in terms of \( v \) and \( y_1 \))? 

**Solution**
Since \( v(x) = y(x) - y_1(x) \), we have

\[
y(x) = v(x) + y_1(x)
\]

and

\[
y'(x) = v'(x) + y'_1(x).
\]

(b) Suppose that \( y_1(x) \) is a particular solution to the Riccati equation

\[
\frac{dy}{dx} = A(x)y^2 + B(x)y + C(x).
\]

Use the formulae found in (a) to make the change of variable \( v = y - y_1 \) and transform this equation into the form of a Bernoulli equation for \( v \).

**Solution**
Since \( y_1(x) \) solves the Riccati equation, it must be that

\[
y'_1 = A(x)y_1^2 + B(x)y_1 + C(x).
\]

Plugging in our substitutions yields

\[
\begin{align*}
v' + y'_1 &= A(x)[v+y_1]^2 + B(x)[v+y_1] + C(x) \\
\Rightarrow v' + [A(x)y_1^2 + B(x)y_1 + C(x)] &= A(x)v^2 + 2A(x)y_1v + A(x)y_1^2 + B(x)v + B(x)y_1 + C(x) \\
\Rightarrow v' &= A(x)v^2 + 2A(x)y_1v + B(x)v \\
\Rightarrow v' + [-2A(x)y_1(x) - B(x)]v &= A(x)v^2.
\end{align*}
\]

This is in the form of a Bernoulli equation.
2. In each of the following problems is a Riccati equation, a function $y_1$ and an initial condition. Verify that the function $y_1$ is a particular solution to the Riccati equation, then find a particular solution satisfying the given initial condition. (Note: you can make the change of variable $v = y - y_1$ to transform the Riccati equation to a Bernoulli equation and solve the resulting Bernoulli equation to obtain all solutions $v = v(x)$. The solutions to the Riccati equation will then be given by $y = v + y_1$.)

(a) $y' = (y - x)^2 + 1; \quad y_1(x) = x; \quad y(0) = \frac{1}{2}$.

**Solution**

First, we verify that $y_1 = x$ is a solution to this equation. Computing, we find that

$$y_1' = 1; \quad \begin{align*}
(y_1 - x)^2 + 1 &= (x - x)^2 + 1 = 1.
\end{align*}
$$

so $y_1 = (y_1 - x)^2 + 1$, so $y_1$ is a solution to the differential equation.

Now we solve the equation:

**Step 1**: Make the change of variables:

substituting $y = v + x$ and $y' = v' + 1$ yields

$$v' + 1 = ((v + x) - x)^2 + 1.$$ 

**Step 2**: Simplify to a Bernoulli equation:

$$v' = v^2.$$ 

Bernoulli equation

(Note that this is also a separable equation and could be solved as such.)

**Step 3**: Solve the Bernoulli equation for $v$.

substep 1: $v = w^{-1}$ and $v' = -w^{-2}w'$, so

$$-w^{-2}w' = (w^{-1})^2$$

substep 2: $w' = -1$.

substep 3: $w = -x + C$.

substep 4: $v = (C - x)^{-1} = \frac{1}{C - x}$.

**General Solution**

substep 5: Yes, $v = 0$ is a solution, and it is singular (not represented in the general solution).

The solutions to the Bernoulli equation are

$$v = \frac{1}{C - x} \quad \text{and} \quad v = 0.$$ 

**Step 4**: Reverse the substitution: $y = v + x$

$$y = \frac{1}{C - x} + x \quad \text{and} \quad y = x.$$ 

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So the solutions are $y = \frac{1}{C-x} + x$ and $y = x$.

Finally, we use the initial condition. The solution $y = x$ can not satisfy the initial condition $y(0) = \frac{1}{2}$, so we use the general solution.

$y(x) = \frac{1}{C-x} + x \Rightarrow y(0) = \frac{1}{C-0} + 0 = \frac{1}{C} = \frac{1}{2} \Rightarrow C = 2.$

(b) $y' = y^2 - \frac{y}{x} - \frac{1}{x^2}, \; x > 0; \; y_1(x) = \frac{1}{x}; \; y(1) = 2.$

**Solution**

First, we verify that $y_1 = \frac{1}{x}$ is a solution to this equation. Computing, we see that

$$y_1^2 - \frac{y_1}{x} - \frac{1}{x^2} = \left(\frac{1}{x}\right)^2 - \left(\frac{1}{x}\right) - \frac{1}{x^2} = \frac{1}{x^2},$$

so $y_1$ is a solution to the differential equation.

Now we solve the equation:

**Step 1:** Make the change of variables:

substituting $y = v + \frac{1}{x}$ and $y' = v' - \frac{1}{x^2}$ yields

$$v' - \frac{1}{x^2} = \left(v + \frac{1}{x}\right)^2 - \frac{1}{x} \left(v + \frac{1}{x}\right) - \frac{1}{x^2}.$$

**Step 2:** Simplify to a Bernoulli equation:

$$v' = v^2 + \frac{2}{x} - \frac{1}{x} \Rightarrow \quad v' - \frac{1}{x} v = v^2.$$  

Bernoulli equation
Step 3: Solve the Bernoulli equation for \( v \).

**substep 1:** \( v = w^{-1} \) and \( v' = -w^{-2}w' \), so
\[
-w^{-2}w' - \frac{1}{x}w^{-1} = (w^{-1})^2
\]

**substep 2:** \( w' + \frac{1}{x}w = -1 \).

**substep 3:** Solve this linear equation for \( w \)

\[
\mu(x) = e^{\int \frac{1}{x} \, dx} = e^{\ln|x|} = e^{\ln x} = x
\]

\[xw' + w = -x \Rightarrow \frac{d}{dx}[xw] = -x
\]

\[\Rightarrow xw = -\frac{1}{2}x^2 + C \Rightarrow w = -\frac{1}{2}x + \frac{C}{x}.
\]

\[\Rightarrow w = \frac{C - x^2}{2x}
\]

**substep 4:** \[v = \left(\frac{C - x^2}{2x}\right)^{-1} = \frac{2x}{C - x^2}.
\]

**substep 5:** Yes, \( v = 0 \) is a solution to the Bernoulli equation, and it is singular (not represented in the general solution).

The solutions to the Bernoulli equation are

\[v = \frac{2x}{C - x^2} \quad \text{and} \quad v = 0.
\]

Step 4: Reverse the substitution:

\[y = v + \frac{1}{x}
\]

\[y = \frac{2x}{C - x^2} + \frac{1}{x} \quad \text{and} \quad y = \frac{1}{x}.
\]

So the solutions are \( y = \frac{2x}{C - x^2} + \frac{1}{x} \) and \( y = \frac{1}{x} \).

Finally, we use the initial condition. The solution \( y = \frac{1}{x} \) can not satisfy the initial condition \( y(1) = 2 \), so we use the general solution.

\[y(x) = \frac{2x}{C - x^2} + \frac{1}{x} \Rightarrow y(1) = \frac{2 \cdot 1}{C - 1^2} + \frac{1}{1} = \frac{2}{C - 1} + 1 = 2 \Rightarrow C = 3.
\]

\[y = \frac{2x}{3 - x^2} + \frac{1}{x}.
\]