Math 240  
Exam 2  
Oct. 22, 2002

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Closed book. You may use a calculator and one 8 ½ × 11" sheet of handwritten notes (both sides). You must show your work to receive full credit. Write solutions in explicit form if possible. All problems have a solution that can be found using the techniques of this class. Problems that ask for answers in complete sentences will be graded on both content and clarity. Points may be deducted for errors in spelling and grammar.

Pledge:  
On my honor, as a student, I have neither given nor received unauthorized aid on this examination: ________________________________ __________________

    (signature)                    (date)
1. Solve the initial value problem,
\[ y'' + 4y' + 5y = 0, \quad y(0) = 1, \quad y'(0) = 0. \]

**First** find the general solution:

Step 1: \((D^2 + 4D + 5)y = 0\)

Step 2: The quadratic formula gives the roots \(-2 \pm i\)

Step 3: The general solution is \(y(x) = c_1 e^{-2x} \cos(x) + c_2 e^{-2x} \sin(x)\)

**Second** solve the initial value problem

We compute
\[ y'(x) = -2c_1 e^{-2x} \cos(x) - c_1 e^{-2x} \sin(x) - 2c_2 e^{-2x} \sin(x) + c_2 e^{-2x} \cos(x). \]
Then plugging in \(x = 0\), we get
\[
\begin{align*}
y(0) &= c_1 = 1 \\
y'(0) &= -2c_1 + c_2 = 0
\end{align*}
\]

Solving the equations, we find \(c_1 = 1\) and \(c_2 = 2\) so the solution to the initial value problem is
\[ y(x) = e^{-2x} \cos(x) + 2e^{-2x} \sin(x) \]
2. Find all solutions to \( y'' + 8y' + 12y = \cos(2x) \). Find all solutions to \( y'' + 8y' + 12y = \cos(2x) \).

Step 1: Solve the homogeneous equation

Substep 1: \((D^2 + 8D + 12)y = 0\)

Substep 2: roots \(-6, -2\)

Substep 3: \(y_h(x) = c_1e^{-6x} + c_2e^{-2x}\)

Step 2: Guess \(y_p(x) = a\cos(2x) + b\sin(2x)\)

Step 3: Plug in and get

\[
y_p'' + 8y_p' + 12y_p = (8a + 16b)\cos(2x) + (-16a + 8b)\sin(2x)
\]

From which we obtain the equations \(8a + 16b = 1\) and \(-16a + 8b = 0\) which we solve to find \(a = \frac{1}{40}\) and \(b = \frac{1}{20}\), so

\(y_p(x) = \frac{1}{40}\cos(2x) + \frac{1}{20}\sin(2x)\).

Step 4: The general solution is

\(y_p(x) = \frac{1}{40}\cos(2x) + \frac{1}{20}\sin(2x) + c_1e^{-6x} + c_2e^{-2x}\)
3. Solve the initial value problem
\[ y'' + 6y' + 9y = e^{-3x}, \quad y(0) = 1, \quad y'(0) = 0. \]

First find the general solution:

Step 1: Find the homogeneous solution
Substep 1: \((D^2 + 6D + 9)y = 0\)
Substep 2: \((D + 3)^2 y = 0\) so we have a double root of \(-3\)
Substep 3: \(y_h = c_1 e^{-3x} + c_2 xe^{-3x}\)

Step 2: Guess \(y_p = Ax^2 e^{-3x}\) since we already have two \(e^{-3x}\) terms in the homogeneous solution.

Step 3: Compute \(y_p'' + 6y_p' + 9y_p = 2Ae^{-3x}\) so \(A = \frac{1}{2}\) and
\[ y_p = \frac{1}{2} x^2 e^{-3x} \]

Step 4: \(y = y_p + y_h = \frac{1}{2} x^2 e^{-3x} + c_1 e^{-3x} + c_2 xe^{-3x}\)

Second plug in the initial values and solve for the constant

We compute \(y' = xe^{-3x} - \frac{3}{2} x^2 e^{-3x} - 3c_1 e^{-3x} - 3c_2 xe^{-3x} + c_2 e^{-3x}\) using the product rule (several times) so \(y(0) = c_1 = 1\) and \(y'(0) = -3c_1 + c_2 = 0\) which we solve to find \(c_1 = 1\) and \(c_2 = 3\). So finally
\[ y = \frac{1}{2} x^2 e^{-3x} + e^{-3x} + 3xe^{-3x} \]
4. Match the following differential equations with the graphs of one of their solutions.

(a) \( y'' + 25y = 30 \cos(5.5t) \)

(b) \( y'' + 25y = 30 \cos(3t) \)

(c) \( y'' + y' + 25y = 30 \cos(5t) \)
5. Approximate $x(1)$ where $x'' + 4x' + \frac{t}{x} = t^2$, $x(0) = 1$, $x'(0) = 1$, using the improved Euler's method with a step-size of $h = 1$. Carry out all calculations to 4 decimal places.

We write this as the system

\[
\begin{align*}
\frac{dx}{dt} &= f(x, y, t) = y, \quad x(0) = 1 \\
\frac{dy}{dt} &= g(x, y, t) = t^2 - 4y - \frac{t}{x}, \quad y(0) = 1
\end{align*}
\]

Then we set $h = 1$ as specified and compute

$x_0 = 1$
$y_0 = 1$
$t_0 = 0$

$f(x_0, y_0, t_0) = 1$

$g(x_0, y_0, t_0) = 0 - 4 \cdot 1 - 0 / 1 = -4$

$\bar{x}_1 = x_0 + hf(x_0, y_0, t_0) = 1 + 1 \cdot 1 = 2$

$\bar{y}_1 = y_0 + hg(x_0, y_0, t_0) = 1 + 1 \cdot (-4) = -3$

$t_1 = t_0 + h = 0 + 1 = 1$

$f(\bar{x}_1, \bar{y}_1, t_1) = -3$

$x(1) = x(t_1) \approx x_1 = x_0 + h \frac{f(x_0, y_0, t_0) + f(\bar{x}_1, \bar{y}_1, t_1)}{2} = 1 + 1 \cdot \frac{1 - 3}{2} = 0$

Note that we don’t need to compute $y_1$ to answer the question posed.
6. Write \( y(t) = \sin(3t) - \sqrt{3} \cos(3t) \) in the form \( A \cos(\omega t + \varphi) \). What is the maximum value of \( y(t) \)?

We write \( \sin(3t) - \sqrt{3} \cos(3t) = \text{Re} \left[ (-\sqrt{3} - i) e^{i3t} \right] \). Then we convert \(-\sqrt{3} - i\) to polar form.

\[
\left| -\sqrt{3} - i \right| = \sqrt{3 + 1} = 2 \quad \text{and} \quad \arg(-\sqrt{3} - i) = \theta \quad \text{where} \quad \tan(\theta) = \frac{-1}{-\sqrt{3}} = \frac{\sqrt{3}}{3}.
\]

Now \( \tan \left( \frac{\pi}{6} \right) = \frac{\sqrt{3}}{3} \) but \( \theta = \frac{\pi}{6} \) is in the first quadrant and \(-\sqrt{3} - i\) is in the fourth quadrant, so we must subtract \( \pi \) radians to get \( \theta = -\frac{5\pi}{6} \) so \(-\sqrt{3} - i = 2e^{-i5\pi/6} \). Hence

\[
\text{Re} \left[ (-\sqrt{3} - i) e^{i3t} \right] = \text{Re} \left[ 2e^{-i5\pi/6} e^{i3t} \right] = \text{Re} \left[ 2e^{i(3t-5\pi/6)} \right] = 2 \cos(3t - 5\pi/6)
\]

Finally, since the amplitude of the cosine wave is 2, the maximum value of \( y(t) \) is 2.
7. Write a paragraph explaining the differences between overdamped and underdamped springs. Your paragraph may include equations and may refer to graphs you draw. In grading the paragraph I will consider both content and clarity and will look for the following specific elements.

(a) What is the difference in behavior of overdamped and underdamped springs (this should be described in words as well as a graph)?

(b) What is the rule for classifying springs as underdamped or overdamped given the mass, damping, and spring constant?

(c) What is the mathematical justification for the rule in (b)?

There are many possible correct answers of course. The following is a sample that hits all the important points.

An underdamped spring bobs up and down repeatedly passing through the equilibrium point while the amplitude decays to 0. An overdamped spring doesn’t oscillate but pulls directly back to equilibrium (possibly passing through equilibrium one time depending on the initial velocity). If a spring-mass system has mass \( m \), damping \( c \), and spring constant \( k \), then the spring is overdamped if

\[ 2ckm > 0 \]

and underdamped if

\[ 2ckm < 0 \] (the case of \( 2ckm = 0 \) is called critically damped). The quantity \( 2ckm \) is the discriminant of the equation

\[ mD^2 + cD + k = 0 \],

so if the system is underdamped the roots are complex conjugates so the solution involves a sine/cosine pair (hence oscillates) while the amplitude decays exponentially. On the other hand, if the system is overdamped, the system has two real roots so the solution is a combination of exponentials which must (eventually) decay monotonically to 0.
8. Suppose a circuit has an inductor, a resistor, and a capacitor (so $L$, $R$, and $C$ are all positive). The power for the circuit comes from a battery that supplies a constant 12 Volts. Show the steady-state current is 0.

The equation for the circuit is $LQ'' + RQ' + \frac{1}{C}Q = 12$. The particular solution takes the form $Q_p = A$ and plugging in we find $Q_p = 12C$. Now we differentiate the charge to get the current. But the derivative of any constant is 0, so the steady-state current must be 0.