Math 240  
Exam 2  
March 25, 2003

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Closed book. You may use a calculator and one 8 ½ × 11" sheet of handwritten notes (both sides). You must show your work to receive full credit. Write solutions in explicit form if possible. All problems have a solution that can be found using the techniques of this class. Problems that ask for answers in complete sentences will be graded on both content and clarity. Points may be deducted for errors in spelling and grammar.

Pledge:  
On my honor, as a student, I have neither given nor received unauthorized aid on this examination:  

(signature)  
(date)
1.

a. Write \( \frac{4-3i}{2-i} \) in the form \( a+bi \).

\[
\frac{4-3i}{2-i} \cdot \frac{2+i}{2+i} = \frac{8+4i-6-3}{4+1} = \frac{11-2i}{5} = \frac{11}{5} - \frac{2}{5}i
\]

b. Write \(-7-2i\) in the form \( re^{i\theta} \). In your answer, you should have \( r > 0 \) and \(-\pi < \theta \leq \pi\).

\[
|-7-2i| = \sqrt{49+4} = \sqrt{53} = 7.28...
\]

\[
\theta = \text{angle}(-7-2i)
\]
\[
\tan \theta = \frac{-2}{-7} = \frac{2}{7}
\]
\[
\arctan \left( \frac{2}{7} \right) = 0.278...
\]

but this is in the first quadrant
and \( \theta \) is in the third quadrant,

so \( \theta = 278... -180 = -2.863... \)

\[
-7-2i = \sqrt{53} e^{i[\arctan(\frac{2}{7}) - \pi]}
\approx 7.28... e^{-2.863...}
2. Solve the initial value problem \( \frac{d^2y}{dx^2} + 8 \frac{dy}{dx} + 25y = 0 \), \( y(0) = 1 \), \( y'(0) = 0 \).

\[
(D^2 + 8D + 25)y = 0
\]

roots: 
\[
-8 \pm \frac{\sqrt{64 - 4 \cdot 1.25}}{2} = \frac{-8 \pm \sqrt{36}}{2} = \frac{-8 \pm 6i}{2} = -4 \pm 3i
\]

\[
y = c_1 e^{-4x} \cos 3x + c_2 e^{-4x} \sin 3x
\]

\[
y' = -4c_1 e^{-4x} \cos 3x - 3c_1 e^{-4x} \sin 3x
\]

\[
y'' = -4c_2 e^{-4x} \sin 3x + 3c_2 e^{-4x} \cos 3x
\]

\[
y(0) = c_1 \quad \text{Set} \quad c_1 = 1
\]

\[
y'(0) = -4c_1 + 3c_2 \quad \text{Set} \quad 3c_2 = 4
\]

\[
c_2 = \frac{4}{3}
\]

\[
y = e^{-4x} \cos(3x) + \frac{4}{3} e^{-4x} \sin(3x)
\]
3. Find the general solution to \( \frac{d^2y}{dx^2} + 3 \frac{dy}{dx} - 10y = e^{2x} \).

Find homosoln.

\[
(D^2 + 3D - 10)y = 0
\]
\[
(0+5)(D+2)
\]

roots \(-5, 2\)

\[
y_h = c_1 e^{-5x} + c_2 e^{2x}
\]

Find part. soln.

Guess \( y_p = A x e^{2x} \) since \( e^{2x} \) already in homosoln.

\[
y_p' = A e^{2x} + 2Axe^{2x}
\]
\[
y_p'' = 2A e^{2x} + 2A e^{2x} + 4Axe^{2x}
\]

\[
y_p'' + 3y_p' - 10y_p = 4A e^{2x} + 4Axe^{2x}
\]
\[
+ 6A e^{2x} + 6Axe^{2x}
\]

\[
-10Axe^{2x}
\]

\[
= \frac{7A e^{2x}}{\frac{e^{2x}}{e^{2x}}}
\]

\[
A = \frac{1}{7}
\]

\[
y = \frac{1}{7} x e^{2x} + c_1 e^{-5x} + c_2 e^{2x}
\]
4. Suppose a circuit has a coil of inductance 30 millihenrys, a resistor of 24 ohms, a capacitor of 1000 microfarads, and a voltage source the produces 200\cos(100t) volts. What is the steady-state current?

\[ 30 \times 10^{-3} \frac{d^2 I}{dt^2} + 24 \frac{d I}{dt} + \frac{1}{1000 \times 10^{-6}} I = \frac{d}{dt} 200 e^{i100t} \]

Guess \( I_p = Ae^{i100t} \), plugging in we get

\[ .03 \cdot (100i)^2 Ae^{i100t} + 24(100i)Ae^{i100t} + \frac{1}{.001} Ae^{i100t} = 200(100i)e^{i100t} \]

Dividing by \((100i)e^{i100t}\)

\[ \left[(.03)(100i)^2 + 24 + \frac{1}{.001} \cdot 100i\right]A = 200 \]

\[ \int 3i + 24 - 10i \int A = 200 \]

\( (24-7i)A = 200 \)

\[ A = \frac{200}{24-7i} \]

Now \( |24-7i| = \sqrt{24^2 + 7^2} = 25 \)

angle \((24-7i) = \arctan(-\frac{7}{24}) \approx -.28379...\)

\[ A = \frac{200}{25e^{-i.28379...}} = 8e^{i.28379...} \]

\( I_p = 8e^{i.28379...} e^{i100t} = 8e^{i(100t + .28379...)} \)

\[ I_{ss} = \text{Re}[I_p] = \frac{8 \cos(100t + .28379...)}{8 \cos(100t + \arctan(\frac{7}{24}))} \]

\[ I_{ss} = \frac{8 \cos(100t + .28379...)}{8 \cos(100t + \arctan(\frac{7}{24}))} \]
5. Match the following differential equations with the graphs of one of their solutions.
(a) \( x'' + 4x = 0 \)
(b) \( x'' + 4x = 2.5\cos(2t) \)
(c) \( x'' + 4x = 5\cos(2.5t) \)

Equation: \( C \)

Equation: \( A \)

Equation: \( B \)
6. Approximate \( x(0.5) \) where \( \frac{d^2x}{dt^2} + x^2 = 0 \), \( x(0)=1 \), \( x'(0)=2 \), using the improved Euler's method with a step-size of \( h=0.5 \). Carry out all calculations to 4 decimal places. **Hint:** first write the second order equation as a first order system.

First write as a first-order system

\[
\frac{dx}{dt} = y \\
\frac{dy}{dt} = \frac{d^2x}{dt^2} = -x^2  \quad (\text{since} \quad \frac{d^2x}{dt^2} + x^2 = 0) \\
y(0) = x'(0) = 2
\]

\[f(x,y,t) = y \]
\[g(x,y,t) = -x^2 \]

\( X_0 = 1 \quad f(x_0, y_0, t_0) = 2 \quad h = 0.5 \)
\( y_0 = 2 \quad g(x_0, y_0, t_0) = -1 \)
\( t_0 = 0 \)

\[\bar{x}_1 = 1 + 0.5 (2) = 2 \quad f(\bar{x}_1, \bar{y}_1, t_1) = 1.5 \]
\[\bar{y}_1 = 2 + 0.5 (-1) = 1.5 \quad g(\bar{x}_1, \bar{y}_1, t_1) = -4 \]
\( t_1 = 0.5 \)

\[
x_1 = 1 + 0.5 \left( \frac{2 + 1.5}{2} \right) = 1.875 \]
\[y_1 = 2 + 0.5 \left( \frac{-1 + -4}{2} \right) = 0.75 \]

\[\sqrt{x(0.5)} \approx 1.875 \]
7. A mass is attached to an undamped spring, causing it to stretch 20cm. The spring is then pulled down another 10cm and released (with initial velocity 0). How long will it take the spring to complete one cycle and return to its initial position (of 10cm down)? **Hint:** Yes you do have enough information to solve this problem. You may find it easiest to write up the equation with an arbitrary mass \( m \) and work with this arbitrary constant (until it drops out).

\[
m \cdot 9.81 = k \cdot 20.
\]

\[
k = \frac{9.81 \, m}{20}
\]

Circular Frequency:

\[
\sqrt{\frac{k}{m}} = \sqrt{\frac{9.81 \, m}{20}} = \sqrt{\frac{9.81}{20}}
\]

\[
\text{Period} = \frac{2\pi}{\sqrt{\frac{9.81}{20}}} \approx 0.897 \ldots \text{ seconds}
\]

It takes one period for the spring to complete one cycle.
8. Given a second order linear differential equation $Ly = f$, if we can find linearly independent $y_1$, $y_2$, and $y_p$ such that $Ly_1 = 0$, $Ly_2 = 0$, and $Ly_p = f$, then the general solution is $y = y_p + c_1y_1 + c_2y_2$. Justify this statement. Your explanation should include:
   (a) the definition of linearity, and
   (b) a calculation that shows $y$ really is a solution to the equation.

A second order differential equation is linear if it can be written in the form
\[ \frac{d^2y}{dx^2} + a(x) \frac{dy}{dx} + b(x)y = f(x). \]
The differential operator $L$ defined by $Ly = \frac{d^2y}{dx^2} + a(x) \frac{dy}{dx} + b(x)y$ is linear, which by definition means

(i) $L(y + z) = Ly + Lz,$

and (ii) $L(cy) = cLy \quad (c \text{ a constant}).$

Applying these rules of linearity, we compute

\[
Ly = L(y_p + c_1y_1 + c_2y_2)
\]
\[
= L(y_p) + L(c_1y_1) + L(c_2y_2)
\]
\[
= L(y_p) + c_1L(y_1) + c_2L(y_2)
\]
\[
= f + c_1 \cdot 0 + c_2 \cdot 0 = f
\]

So $y$ is a solution to the equation with 2 arbitrary constants, hence it is the general solution.