1. Let $f(t)$ be the “impulse train” $f(t) = \delta(t) + \delta(t - \pi) + \delta(t - 2\pi) + \cdots$. Find the Laplace transform of the solution to the initial value problem

$$x'' + 4x = f(t), \quad x(0) = 0, \quad x'(0) = 0.$$  

Show that the Laplace transform of the solution has a double pole on the Imaginary axis, which suggests (but doesn’t prove) that the solution will be in resonance. Explain why resonance is reasonable here.

2. When we first considered population models, we worked with just a single population. More reasonable models often deal with several interacting species, which leads to systems of equations. For example, a predator-prey model is a model of two species, one of which is a predator on the other. Let the population of the prey species as a function of time be $x(t)$ and the population of the predator species be $y(t)$. Then a typical example of a predator-prey system is

$$\frac{dx}{dt} = 100x - 5xy$$
$$\frac{dy}{dt} = -10y + 2xy.$$  

Unfortunately, this is a non-linear, autonomous system (it is non-linear since it involves products of the two dependent variables, $x$ and $y$, and autonomous since the right-hand side doesn’t involve the independent variable $t$). Since it is non-linear we can’t simplify using Laplace transforms. Fortunately, since it is autonomous, there is another trick available. Using the fact that $\frac{dy}{dx} = \frac{dy/\,dt}{dx/\,dt}$, rewrite this autonomous system as a first-order equation and then solve the equation to find a formula relating the populations $x$ and $y$. Note that while this equation will tell you how $x$ and $y$ are related, it won’t tell you how either depends on $t$. 