Written Assignment #2:
Exact and Linear Equations
Due 5:00pm Tuesday, September 9, 2003

You are encouraged to collaborate with your colleagues. For credit, however, your final write-up must be done individually. Show all your work and make your presentation comprehensible.

1. (a) Show that the following differential equation is not exact:

\[ y + (2t - ye^y) \frac{dy}{dt} = 0. \]

(b) Show that if you multiply the above differential equation by \( y \), it becomes exact; then find an implicit solution satisfying \( y(2) = 1 \).

2. In this problem, you will show that if \( p(x) \) and \( q(x) \) are continuous functions for all \( x \), then there is at most one function \( y = y(x) \) satisfying the initial value problem

\[ y' + p(x)y = q(x); \quad y(0) = y_0. \]  \hspace{1cm} (1)

It might be helpful to review the beginning of Section 1.5 in the text.

(a) Suppose that \( y_1(x) \) and \( y_2(x) \) are both solutions to the differential equation in (1). Verify that the function \( v = v(x) = y_1(x) - y_2(x) \) solves the initial value problem

\[ v' + p(x)v = 0; \quad v(0) = 0. \]

(b) With \( \mu(x) = e^{\int p(x) \, dx} \), show that

\[ \frac{d}{dx} \left[ \mu(x) [y_1(x) - y_2(x)] \right] = 0. \]

Conclude that \( \mu(x) [y_1(x) - y_2(x)] \) is constant.

(Hint: what does \( \frac{d}{dx} \left[ \mu(x) v(x) \right] \) equal?)

(c) From part (a), we have that \( y_1(0) - y_2(0) = 0 \). Also, \( \mu(x) = e^{\int p(x) \, dx} > 0 \) for all values of \( x \) (i.e. \( \mu(x) \) is never zero). Use part (b), to argue that \( y_1(x) - y_2(x) = 0 \) for all values of \( x \). This implies that \( y_1(x) = y_2(x) \) for all \( x \), and there cannot be two distinct solutions to the linear differential equation in (1); in other words, there is at most one solution.