Initial Value Problems (Section 1.3):
Solutions for Selected Problems

(1) \[ \frac{dy}{dx} = y^2 - 9; \quad y(0) = 1. \]

First: Find all solutions.

Step 1: \[ \frac{1}{(y - 3)(y + 3)} \, dy = dx. \]

Step 2: \[ \int \left( \frac{1}{6 \, y - 3} - \frac{1}{6 \, y + 3} \right) \, dy = x + C \Rightarrow \frac{1}{6} \ln |y - 3| - \frac{1}{6} \ln |y + 3| = x + C. \]

Step 3: \[ \ln \left| \frac{y - 3}{y + 3} \right| = 6x + C \Rightarrow \frac{y - 3}{y + 3} = Ce^{6x} \]

\[ \Rightarrow y = \frac{3 \left( 1 + Ce^{6x} \right)}{1 - Ce^{6x}}. \]

General Solution

Step 4: In Step 1, we divided by \((y - 3)(y + 3)\)

\[ y(x) = 3 \left\{ \begin{array}{l} \text{Is } y = y(x) = 3 \text{ a solution to the DiffEq?} \\
\text{Yes, it is.} \\
\text{Is the solution } y(x) = 3 \text{ represented in the general solution?} \\
\text{Yes, it is (use } C = 0 \text{).} \\
\text{Is } y = y(x) = -3 \text{ a solution to the DiffEq?} \\
\text{Yes, it is.} \\
\text{Is the solution } y(x) = -3 \text{ represented in the general solution?} \\
\text{No, it is not represented.} \\
\end{array} \right. \]

\[ y(x) = 3 \text{ is not a singular solution;} \]

\[ y(x) = -3 \text{ is a singular solution.} \]

The solutions are \( y = \frac{3 \left( 1 + Ce^{6x} \right)}{1 - Ce^{6x}} \) and \( y = -3 \).

Second: Use initial value (notice that the singular solution doesn’t satisfy the initial condition in this problem).

\[ y(x) = 3 \left( \frac{1 + Ce^{6x}}{1 - Ce^{6x}} \right) \Rightarrow y(0) = 3 \left( \frac{1 + Ce^{0}}{1 - Ce^{0}} \right) = 1 \Rightarrow 3 + 3C = 1 - C \Rightarrow C = -\frac{1}{2}. \]

\[ y = \frac{3 \left( 1 - \frac{1}{2} e^{6x} \right)}{1 + \frac{1}{2} e^{6x}}. \]
(3) \( \frac{dy}{dx} = xy - 3x; \ y(0) = 0. \)

First: Find all solutions.

Step 1: \( \frac{1}{y-3} \) \( dy = x \) \( dx. \)

Step 2: \( \ln |y-3| = \frac{1}{2} x^2 + C. \)

Step 3: \( y = Ce^{\frac{1}{2}x^2} + 3. \)

General Solution

Step 4: In Step 1, we divided by \( y-3 \)

Is \( y = y(x) = 3 \) a solution to the DiffEq?

Yes, it is.

Is the solution \( y(x) = 3 \) represented in the general solution?

\( y = Ce^{\frac{1}{2}x^2} + 3 = 3 \Rightarrow C = 0. \)

Yes, it is (use \( C = 0 \))

\( y(x) = 3 \) is not a singular solution.

The solutions are \( y = Ce^{\frac{1}{2}x^2} + 3. \)

Second: Use the initial value.

\( y(x) = Ce^{\frac{1}{2}x^2} + 3 \Rightarrow y(0) = Ce^{\frac{1}{2}0^2} + 3 = 0 \Rightarrow C + 3 = 0 \Rightarrow C = -3. \)

\( y = 3 - 3e^{\frac{1}{2}x^2}. \)

(5) \( \frac{dy}{dx} = xy - x - y + 1; \ y(3) = -3. \)

First: Find all solutions.

Step 1: \( \frac{dy}{dx} = x(y - 1) - (y - 1) \Rightarrow \frac{1}{y - 1} \) \( dy = (x - 1) \) \( dx. \)

Step 2: \( \ln |y - 1| = \frac{1}{2} x^2 - x + C \)

Step 3: \( y = Ce^{\frac{1}{2}x^2-x} + 1. \)

General Solution

Step 4: In Step 1, we divided by \( y - 1 \)

Is \( y = y(x) = 1 \) a solution to the DiffEq?

Yes, it is.

Is the solution \( y(x) = 1 \) represented in the general solution?

\( y = Ce^{\frac{1}{2}x^2-x} + 1 = 1 \Rightarrow C = 0. \)

Yes, it is (use \( C = 0 \))

\( y(x) = 1 \) is not a singular solution.

The solutions are \( y = Ce^{\frac{1}{2}x^2-x} + 1. \) Second: Use the initial value.

\( y(x) = Ce^{\frac{1}{2}x^2-x} + 1 \Rightarrow y(3) = Ce^{\frac{1}{2}3^2-3} + 1 = -3 \Rightarrow C e^{\frac{3}{2}} = -4 \Rightarrow C = -4e^{-\frac{3}{2}} \)

\( y = -4e^{-\frac{3}{2}} e^{\frac{1}{2}x^2-x} + 1 = -4e^{\frac{1}{2}x^2-x-\frac{3}{2}} + 1. \)
(7) \[ \frac{dy}{dx} = y^2 - 5y + 6; \quad y(0) = 2. \]

First: Find all solutions.

Step 1: \[ \frac{1}{(y - 3)(y - 2)} \, dy = dx. \]

Step 2: \[ \ln |y - 3| - \ln |y - 2| = x + C. \]

Step 3: \[ \frac{y - 3}{y - 2} = Ce^x \Rightarrow y = \frac{3 - 2Ce^x}{1 - Ce^x} \]

General Solution

it’s the same \( C \) in the numerator and denominator, so we can not replace \( 2C \) with \( C \) in the general solution.

Step 4: In Step 1, we divided by \( (y - 3)(y - 2) \)

\[
\begin{align*}
\text{Is } y &= y(x) = 3 \text{ a solution to the DiffEq?} \\
\text{Yes, it is.} \\
\text{Is the solution } y(x) = 3 \text{ represented in the general solution?} \\
y &= \frac{3 - 2Ce^x}{1 - Ce^x} = 3 \Rightarrow Ce^x = 0 \Rightarrow C = 0. \\
\text{Yes, it is (use } C = 0) \\
\text{Is } y &= y(x) = 2 \text{ a solution to the DiffEq?} \\
\text{Yes, it is.} \\
\text{Is the solution } y(x) = 2 \text{ represented in the general solution?} \\
y &= \frac{3 - 2Ce^x}{1 - Ce^x} = 2 \Rightarrow 3 = 2 \text{ (no good).} \\
\text{No, it is not represented.} \\
y(x) &= 3 \text{ is not a singular solution; } \\
y(x) &= 2 \text{ is a singular solution.}
\end{align*}
\]

The solutions are \( y = \frac{3 - 2Ce^x}{1 - Ce^x} \) and \( y = 2 \).

Second: Use the initial value. For this problem the singular solution \( y = 2 \) does satisfy the initial condition; this is one answer to the problem. We should still check if there is a way to pick \( C \) in the general solution so that the intial condition is satisfied.

\[ y(x) = \frac{3 - 2Ce^x}{1 - Ce^x} \Rightarrow y(0) = \frac{3 - 2Ce^0}{1 - Ce^0} = 2 \Rightarrow 3 - 2C = 2 - 2C \Rightarrow 3 = 2 \text{ (no good).} \]

\[ y = 2. \]

(9) \[ \frac{dy}{dx} = xy^2 - 8x - 2xy; \quad y(1) = -2. \]

First: Find all solutions.

Step 1: \[ \frac{dy}{dx} = x(y^2 - 2y - 8) \Rightarrow \frac{1}{(y - 4)(y + 2)} \, dy = x \, dx. \]

Step 2: \[ \frac{1}{6} \ln |y - 4| - \frac{1}{6} \ln |y + 2| = \frac{1}{2} x^2 + C. \]

Step 3: \[ \frac{y - 4}{y + 2} = Ce^{3x^2} \Rightarrow y = \frac{4 + 2Ce^{3x^2}}{1 - Ce^{3x^2}} \]

General Solution

as in the previous problem, we can not replace \( 2C \) with \( C \).
Step 4: In Step 1, we divided by \((y - 4)(y + 2)\)

\[
y(x) = 4\begin{cases} 
\text{Is } y = y(x) = 4 \text{ a solution to the DiffEq?} \\
\text{Yes, it is.} \\
\text{Is the solution } y(x) = 4 \text{ represented in the general solution?} \\
y = \frac{4 + 2Ce^{3x^2}}{1 - Ce^{3x^2}} = 4 \Rightarrow 6Ce^{3x^2} = 0 \Rightarrow C = 0. \\
\text{Yes, it is (use } C = 0) \\
\text{Is } y = y(x) = -2 \text{ a solution to the DiffEq?} \\
\text{Yes, it is.} \\
\text{Is the solution } y(x) = 2 \text{ represented in the general solution?} \\
y = \frac{4 + 2Ce^{3x^2}}{1 - Ce^{3x^2}} = -2 \Rightarrow 4 = -2 \text{ (no good).} \\
\text{No, it is not represented.} \\
y(x) = 4 \text{ is not a singular solution;} \\
y(x) = -2 \text{ is a singular solution.}
\]

The solutions are \(y = -2\) and \(y = \frac{4 + 2Ce^{3x^2}}{1 - Ce^{3x^2}}\).

Second: Use the initial value. For this problem the singular solution \(y = -2\) does satisfy the initial condition; this is one answer to the problem. We will still check if there is a way to pick \(C\) in the general solution so that the intial condition is satisfied.

\[
y(x) = \frac{4 + 2Ce^{3x^2}}{1 - Ce^{3x^2}} \Rightarrow y(1) = \frac{4 + 2Ce^{3-1^2}}{1 - Ce^{3-1^2}} = -2 \\
\Rightarrow 4 + 2e^3C = -2 + 2e^3C \Rightarrow 4 = -2 \text{ (no good).}
\]

\[
y = -2.
\]

(11) \(\frac{dy}{dx} = \frac{x}{y}; 
\quad y(0) = 1.\)

First: Find all solutions

Step 1: \(y dy = x dx.\)

Step 2: \(y^2 = x^2 + C.\)

Step 3: \(y = \pm \sqrt{x^2 + C}.\)

General Solution

\[\text{Step 4: In Step 1, we did not divided by } \frac{1}{y}, \text{ which is never zero, so there are no singular solutions.}\]

The solutions are \(y = \pm \sqrt{x^2 + C}.\)

Second: Use the initial value.

\[
y(x) = \pm \sqrt{x^2 + C} \Rightarrow y(0) = \pm \sqrt{0^2 + C} = 1 \\
\Rightarrow C = 1 \text{ and we use the positive root.}
\]

\[
y = \sqrt{x^2 + 1}.
\]
(13) \( \frac{dy}{dx} = \cos^2 y; \quad y(0) = \frac{\pi}{2}. \)

First: Find all solutions

Step 1: \( \sec^2 y \, dy = dx. \)

Step 2: \( \tan y = x + C. \)

Step 3: \( y = \arctan(x + C). \)

Step 4: In Step 1, we divided by \( \cos^2 y, \) which is zero at every odd multiple of \( \frac{\pi}{2}, \) i.e. \( (2k + 1)\frac{\pi}{2} \) with \( k \) an integer.

Is \( y = y(x) = (2k + 1)\frac{\pi}{2} \) a solution to the DiffEq?

Yes. For each integer \( k, \) it is.

Is the solution \( y(x) = (2k + 1)\frac{\pi}{2} \) represented in the general solution?

\[
y = \arctan(x + C) = (2k + 1)\frac{\pi}{2} \Rightarrow x + C = \tan\left((2k + 1)\frac{\pi}{2}\right)
\]

(no good, since the tangent function is not defined at odd multiples of \( \frac{\pi}{2}. \))

For each integer \( k, \) \( y(x) = (2k + 1)\frac{\pi}{2} \) is a singular solution.

The solutions are \( y = \arctan(x + C) \) and \( y = (2k + 1)\frac{\pi}{2}, \) with \( k \) an integer.

Second: Use the initial condition. The singular solution \( y = \frac{\pi}{2} \) satisfies the initial condition, so this is one solution to the problem. We will see if there are any others from the general solution.

\[
y(x) = \arctan(x + C) \Rightarrow y(0) = \arctan(0 + C) = \frac{\pi}{2}
\]

\[
\Rightarrow C = \tan \frac{\pi}{2} \quad \text{(no good; tan} \frac{\pi}{2} \text{is not defined)}.
\]

\[
y = \frac{\pi}{2}.
\]

(15) \( \frac{dy}{dx} = x^2 + 4y; \quad y(0) = 1. \)

First: Find all solutions

Step 1: \( \frac{1}{y} \, dy = (x^2 + 4) \, ds. \)

Step 2: \( \ln |y| = \frac{1}{3}x^3 + 4x + C. \)

Step 3: \( y = Ce^{\left(\frac{1}{3}x^3 + 4x\right)}. \)

Step 4: In Step 1, we divided by \( y \)

Is \( y = y(x) = 0 \) a solution to the DiffEq?

Yes, it is.

Is the solution \( y(x) = 0 \) represented in the general solution?

\[
y = Ce^{\left(\frac{1}{3}x^3 + 4x\right)} = 0 \Rightarrow C = 0.
\]

Yes, it is (use \( C = 0 \))

\( y(x) = 0 \) is not a singular solution.
The solutions are $y = C e^{\left(\frac{1}{3} x^3 + 4x\right)}$.

Second: Use the initial condition.

$$y(x) = C e^{\left(\frac{1}{3} x^3 + 4x\right)} \Rightarrow y(0) = C = 1.$$  

(17) $\frac{dy}{dx} = e^{x-y}; \quad y(0) = 1$. 

First: Find all solutions

Step 1: $e^y \frac{dy}{dx} = e^x \; dx$.  
Step 2: $e^y = e^x + C$.  
Step 3: $y = \ln\left(e^x + C\right)$.

General Solution

Step 4: In Step 1, we divided by $e^{-y}$, which is never zero, so there are no singular solutions.

The solutions are $y = \ln(e^x + C)$. Second: Use the initial condition.

$$y(x) = \ln(e^x + C) \Rightarrow y(0) = \ln(1 + C) = 1 \Rightarrow C = e - 1.$$  

(19) $\frac{dy}{dx} = xy^2 - x + 2y^2 - 2; \quad y(0) = 1$. 

First: Find all solutions.

Step 1: $\frac{1}{(y-1)(y+1)} \; dy = (x + 2) \; dx$.  
Step 2: $\frac{1}{2} \ln |y-1| - \frac{1}{2} \ln |y+1| = \frac{1}{2} x^2 + 2x + C$.  
Step 3: $\frac{y - 1}{y + 1} = Ce^{(x^2 + 4x)} \Rightarrow y = \frac{1 + Ce^{(x^2 + 4x)}}{1 - Ce^{(x^2 + 4x)}}$.

General Solution

Step 4: In Step 1, we divided by $(y - 1)(y + 1)$

$$y(x) = 1 \begin{cases} 
\text{Is } y = y(x) = 1 \text{ a solution to the DiffEq?} \\
\text{Yes, it is.} \\
\text{Is the solution } y(x) = 1 \text{ represented in the general solution?} \\
\frac{1 + Ce^{(x^2 + 4x)}}{1 - Ce^{(x^2 + 4x)}} = 1 \Rightarrow 2Ce^{(x^2 + 4x)} = 0 \Rightarrow C = 0. \\
\text{Yes, it is (use } C = 0) \\
\text{Is } y = y(x) = -1 \text{ a solution to the DiffEq?} \\
\text{Yes, it is.} \\
\text{Is the solution } y(x) = -1 \text{ represented in the general solution?} \\
\frac{1 + Ce^{(x^2 + 4x)}}{1 - Ce^{(x^2 + 4x)}} = -1 \Rightarrow 1 = -1 \text{ (no good).} \\
\text{No, it is not represented.} \\
\end{cases}$$ 

$y(x) = 1$ is not a singular solution; $y(x) = -1$ is a singular solution.
The solutions are $y = \frac{1 + Ce^{(x^2+4x)}}{1 - Ce^{(x^2+4x)}}$ and $y = -1$.

Second: Use the initial condition. The singular solutions can not satisfy the initial conditions.

$$y(x) = \frac{1 + Ce^{(x^2+4x)}}{1 - Ce^{(x^2+4x)}} \Rightarrow y(0) = \frac{1 + C}{1 - C} = 1 \Rightarrow C = 0.$$ 

$y = 1.$