Written Assignment #1: Separable Equations (Solutions)

1. Find \( y(1) \) if \( \frac{dy}{dx} = \frac{4e^x}{2y + 2} \) and \( y(0) = -3 \).

**Solution**

First: Find all solutions.

\[
\int (2y + 2) \, dy = \int 4e^x \, dx \Rightarrow y^2 + 2y = 4e^x + C
\]

\[\Rightarrow (y + 1)^2 - 1 = 4e^x + C \Rightarrow y = -1 \pm \sqrt{4e^x + C} \]

Second: Use the initial condition to determine the value for \( C \) and which root to use.

\[y(x) = -1 \pm \sqrt{4e^x + C} \Rightarrow y(0) = -1 \pm \sqrt{4e^0 + C} = -3 \]

\[\Rightarrow 4 + C = 4 \Rightarrow C = 0.\]

We now know that \( y(x) = -1 \pm \sqrt{4e^x} = -1 \pm 2e^{\frac{1}{2}x} \), but we still need to decide which root to use. Which one satisfies the initial condition?

\[y(x) = -1 + 2e^{\frac{1}{2}x} \Rightarrow y(0) = -1 + 2 = 1 \neq -3 \text{ (no good).} \]

\[y(x) = -1 - 2e^{\frac{1}{2}x} \Rightarrow y(0) = -1 - 2 = -3 \text{ (this is good).} \]

The solution we want is \( y(x) = -1 - 2e^{\frac{1}{2}x} \). Now, we can compute

\[y(1) = -1 - 2e^{\frac{1}{2}}.\]

2. (a) Newton’s Law of Cooling (or Heating) provides a simple model for the change in an object’s temperature in a surrounding environment with a constant temperature. In words, the law states that the instantaneous rate of change of the temperature of an object is proportional to the difference in temperature between the object and its surrounding environment. Write a differential equation that expresses Newton’s Law of Cooling mathematically. Use \( k > 0 \) for the constant of proportionality, \( \beta \) for the constant temperature of the surrounding environment, \( T \) for the temperature of the object, and \( t \) for time.
**Solution**

Mathematically, the instantaneous rate of change in \( T \) (temperature) with respect to \( t \) (time) is \( \frac{dT}{dt} \). Newton’s Law of Cooling states that this should be proportional to \( (T - \beta) \) (difference in the object’s and environment’s temperature). So the differential equations is

\[
\frac{dT}{dt} = -k(T - \beta).
\]

How did we decide to use a \( - \) sign in front of \( k \)? We ask ourselves what happens if the object’s temperature is higher than the temperature of the surrounding environment \( (T > \beta) \). The object should cool off; i.e. its temperature should decrease; i.e. \( \frac{dT}{dt} \) should be negative. Since \( k > 0 \) and \( T - \beta > 0 \), in this situation \( k(T - \beta) \) is positive, and we must use the negative of it in our differential equation.

(b) Suppose that the constant of proportionality in Newton’s Law is .08/hr for coffee. In a room with a constant temperature of 60° F, the temperature of a cup of coffee is measured to be 200° F. What is temperature of the coffee 3 hours later?

**Solution**

We first decide for what value of \( t \) is the coffee 200° F. We may as well say that this occurs when \( t = 0 \). Plugging in numbers, here is the mathematical problem to solve

\[
\frac{dT}{dt} = -0.08(T - 60); \quad T(0) = 200.
\]

Now, we solve this initial value problem.

\[
\int \frac{1}{T - 60} dT = \int (-0.08) dt \Rightarrow \ln |T - 60| = -0.08t + C \Rightarrow T = Ce^{-0.08t} + 60.
\]

Using the initial condition, we find that

\[
T(0) = C + 60 = 200 \Rightarrow C = 140.
\]

Thus

\[
T(t) = 140e^{-0.08t} + 60.
\]
Since we used $k = .08/\text{hr}$, time is measured in hours. The temperature of the coffee after 3 hours is the value of $T(3)$

$$T(3) = 140e^{-.24} + 60 \approx 170.1^\circ \text{F}.$$  

(c) A forensic specialist measured the temperature of a murder victim’s body at 11:00pm and found it to be 90.1\degree F. At 11:30pm, the temperature was 89.5\degree F. Assuming the victim’s normal body temperature was 98.6\degree F, at what time was the murder committed if the temperature of the room was a constant 70\degree F?

**Solution**

For this problem, we are not given the constant of proportionality; we have to determine it from the information given. First, we need to decide when $t = 0$ occurs and our units for time. Let say that $t = 0$ at 11:00pm, and we will measure time in hours. Mathematically, the information that we have is

$$\frac{dT}{dt} = -k(T - 70); \quad T(0) = 90.1 \text{ and } T(.5) = 89.5.$$  

We want to know when did $T = 98.6$, this will be the time of death. First we solve the differential equation (remember that $k$ is a constant; it does not change with $t$).

$$\int \frac{1}{T - 70} \, dT = \int (-k) \, dt \Rightarrow \ln |T - 70| = -kt + C$$  

$$\Rightarrow T = Ce^{-kt} + 70.$$  

There are two unknowns in this equation, so we need to find two equations

Equation 1: $T(0) = 90.1 \Rightarrow C + 70 = 90.1$  
Equation 2: $T(.5) = 89.5 \Rightarrow Ce^{-.5k} + 70 = 89.5$.

From the Equation 1, we see that $C = 20.1$, plugging this into Equation 2 yields

$$20.1e^{-.5k} + 70 = 89.5 \Rightarrow e^{-.5k} = \frac{19.5}{20.1} \Rightarrow -.5k = \ln \left( \frac{19.5}{20.1} \right)$$  

$$\Rightarrow k = -2 \ln \left( \frac{19.5}{20.1} \right).$$  

We now have $C$ and $k$, putting them into the formula for $T$ gives us

$$T(t) = (20.1)e^{2\ln\left( \frac{19.5}{20.1} \right)t} + 70.$$  

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For what value of $t$ is $T$ equal to 98.6?

$$(20.1)e^{2\ln\left(\frac{19.5}{20.1}\right)t} + 70 = 98.6 \Rightarrow e^{2\ln\left(\frac{19.5}{20.1}\right)t} = \frac{28.6}{20.1}$$

$$\Rightarrow t = \frac{\ln\left(\frac{28.6}{20.1}\right)}{2\ln\left(\frac{19.5}{20.1}\right)} \approx -5.819 \text{ hours}.$$  

The victim’s body had a temperature of 98.6°F 5 hours and 49 minutes before 11:00pm, so the murder was committed at about 5:11pm.