Written Assignment #13: Power Series and Radius of Convergence (solutions)

1. Determine the radius of convergence for each of the following power series.

(a) \( \sum_{n=1}^{\infty} \frac{x^{3n+2}}{n} \).

Solution
We use the ratio test to establish the radius of convergence. The series is absolutely convergent provided that
\[
\lim_{n \to \infty} \left| \frac{x^{3(n+1)+2}}{n+1} / \frac{x^{3n+2}}{n} \right| < 1.
\]
Computing the limit, we find that we need
\[
\lim_{n \to \infty} \left| \frac{n+1}{n} \cdot \frac{x^{2(n+1)}}{x^{2n}} \right| = \lim_{n \to \infty} \left| x^2 \right| \lim_{n \to \infty} \left| \frac{n+1}{n} \right| = \left| x^2 \right| \cdot 1 < 1.
\]
This inequality is true whenever \( |x - 0| < 1 \). Thus, the radius of convergence is 1. This means that the series converges whenever \(-1 < x < 1\) and diverges whenever \(|x| > 1\).

(b) \( \sum_{n=0}^{\infty} \frac{n}{4^n} x^{2n} \).

Solution
The ratio test tells us that this series converges absolutely provided that
\[
\lim_{n \to \infty} \left| \frac{n+1}{4n} \cdot \frac{x^{2(n+1)}}{x^{2n}} \right| < 1.
\]
Computing the limit, we find that we need
\[
\lim_{n \to \infty} \left| \frac{n+1}{4n} \cdot x^2 \right| \left( \frac{n+1}{4n} \right) = \left| x^2 \right| \lim_{n \to \infty} \left( \frac{n+1}{4n} \right) = \left| x \right| \cdot \frac{1}{4} < 1.
\]
This inequality is true whenever \(|x - 0| < 2\). Thus the radius of convergence is 2. This means that the series converges whenever \(-2 < x < 2\) and diverges whenever \(|x| > 2\).
(c) \( \sum_{n=1}^{\infty} \frac{(x-x_0)^n}{n} \), where \( x_0 \) is a given number.

**Solution**

This is a power series centered at \( x_0 \). We can again use the root test. The series converges absolutely provided that

\[
\lim_{n \to \infty} \left| \frac{(x-x_0)^{n+1}}{n+1} \right| < 1.
\]

Evaluating the limit tells us that we need

\[
\lim_{n \to \infty} \left| (x-x_0) \frac{n}{n+1} \right| = |x-x_0| \lim_{n \to \infty} \frac{n}{n+1} = |x-x_0| \cdot 1 < 1.
\]

This inequality is true whenever \( |x-x_0| < 1 \). Thus, the radius of convergence is 1. This means that the series converges whenever \( x_0 - 1 < x < x_0 + 1 \) and diverges whenever \( |x-x_0| > 1 \).

2. Determine the Taylor series about the point \( x_0 \) for each of the following functions. Also determine the radius of convergence.

(a) \( \frac{x}{2-x}, \quad x_0 = 0 \).

**Solution**

Recall that \( \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \) (provided that \( |z| < 1 \)). We may write

\[
\frac{x}{2-x} = \frac{x}{2} \frac{1}{1 - \left[ \frac{1}{2}x \right]} = \frac{x}{2} \sum_{n=0}^{\infty} \left( \frac{1}{2}x \right)^n = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} x^{n+1}.
\]

We can adjust the index of summation to get the power series

\[
\sum_{n=1}^{\infty} \frac{1}{2^n} x^n.
\]

Using the ratio test, we find that this series converges absolutely provided that

\[
\lim_{n \to \infty} \left| \frac{\frac{1}{2^{n+1}} x^{n+1}}{\frac{1}{2^n} x^n} \right| = |x| \cdot \frac{1}{2} < 1.
\]

Thus the series converges whenever \( |x-0| < 2 \) and the radius of convergence is 2.
(b) \( \frac{x}{2 - x}, \quad x_0 = 5. \)

**Solution**

We may write

\[
\frac{x}{2 - x} = \frac{x}{2 - (x - 5) - 5} = -\frac{x}{3} \cdot \frac{1}{1 - \frac{1}{3}(x - 5)} = \frac{x}{3} \sum_{n=0}^{\infty} \left[ -\frac{1}{3}(x - 5) \right]^n.
\]

This is not quite in the proper form of a power series, so we need to manipulate it further.

\[
\frac{x}{2 - x} = \frac{1}{3}(x - 5 + 5) \sum_{n=0}^{\infty} \left( -\frac{1}{3} \right)^n (x - 5)^n
\]

\[
= \sum_{n=0}^{\infty} \left( -\frac{1}{3} \right)^{n+1} (x - 5)^{n+1} + \sum_{n=0}^{\infty} 5 \left( -\frac{1}{3} \right)^{n+1} (x - 5)^n
\]

\[
= \sum_{n=1}^{\infty} \left( -\frac{1}{3} \right)^n (x - 5)^n + \sum_{n=1}^{\infty} 5 \left( -\frac{1}{3} \right)^{n+1} (x - 5)^n - \frac{5}{3}.
\]

We now combine these two series to get the power series

\[
-\frac{5}{3} + \sum_{n=1}^{\infty} \left[ \left( -\frac{1}{3} \right)^n \left( 1 - \frac{5}{3} \right) \right] (x - 5)^n.
\]

Using the ratio test, we find that this series converges absolutely provided that

\[
\lim_{n \to \infty} \left| \left( -\frac{1}{3} \right)^{n+1} \left( 1 - \frac{5}{3} \right) \right| \left| (x - 5)^{n+1} \right| / \left| \left( -\frac{1}{3} \right)^n \left( 1 - \frac{5}{3} \right) \right| \left| (x - 5)^n \right| = \lim_{n \to \infty} \left| (x - 5) \left( -\frac{1}{3} \right) \right| = |x - 5| \cdot \frac{1}{3} < 1.
\]

Thus, the series converges whenever \(|x - 5| < 3\) and the radius of convergence is 3. This means that this series converges whenever \(2 < x < 8\) and diverges whenever \(|x - 5| > 3\).