Written Assignment #6: Linear Operators (solutions)

1. Determine whether or not the following operators are linear.

(a) $Ly = x^2 D^2 y - 2y$

**Solution**

We check if both of the two conditions for a linear operator are satisfied. For the 1st condition, we will see if $L[f + g] = Lf + Lg$ for any functions $f$ and $g$.

$L[f + g] = x^2 D^2 [f + g] - 2[f + g] = x^2 \frac{d^2}{dx^2} [f + g] - 2f - 2g$

$= x^2 \left[ \frac{d^2}{dx^2} f + \frac{d^2}{dx^2} g \right] - 2f - 2g = x^2 D^2 f + x^2 D^2 g - 2f - 2g$

$= \{ x^2 D^2 f - 2f \} + \{ x^2 D^2 g - 2g \} = Lf + Lg.$

So the 1st condition is satisfied. Now we check the 2nd condition: does $L[cf] = cLf$ for all constants $c$ and functions $f$?

$L[cf] = x^2 D^2 [cf] - 2[cf] = x^2 c D^2 f - c2f = c \{ x^2 D^2 f - 2f \}$

$= cLf.$

Thus $L$ is a linear operator.

(b) $Ly = y D^2 y + (Dy)^2$;

**Solution**

Again, we check if the conditions for $L$ to be a linear operator are satisfied. First, we see if $L[f + g] = Lf + Lg$.

$L[f + g] = [f + g] D^2 [f + g] + (D[f + g])^2$

$= (f + g) (D^2 f + D^2 g) + (Df + Dg)^2$

$= f D^2 f + f D^2 g + g D^2 f + g D^2 g + (Df)^2 + 2(Df)(Dg) + (Dg)^2$

$= \{ f D^2 f + (Df)^2 \} + \{ g D^2 g + (Dg)^2 \} + f D^2 g + g D^2 f + 2(Df)(Dg)$

$= Lf + Lg + f D^2 g + g D^2 f + 2(Df)(Dg)$

$\neq Lf + Lg$ in general.

Since $L$ fails to satisfy the first condition, there is no need to check the second. The operator $L$ is not linear.
2. (a) With \( Ly = x^2 D^2 y - 2y \), write \( Ly = 0 \) as a differential equation. Verify that
\[
y_1(x) = x^2 \quad \text{and} \quad y_2(x) = \frac{1}{x}
\]
are solutions to this differential equation, for \( x > 0 \). If \( C_1 \) and \( C_2 \) are constants, is the linear combination \( C_1 y_1(x) + C_2 y_2(x) \) a solution to this differential equation, for \( x > 0 \)?

**Solution**

The differential equation \( Ly = 0 \) is
\[
x^2 \frac{d^2 y}{dx^2} - 2y = 0.
\]

Now, we check if \( y_1 \) is a solution. Noting that \( y_1'' = 2 \), we have
\[
x^2 y_1'' - 2y_1 = x^2 (2) - 2 \left( x^2 \right) = 0.
\]
The function \( y_1 \) is a solution.

Next, we check if \( y_2 \) is a solution. Noting that \( y_2'' = \frac{2}{x^3} \), we have
\[
x^2 y_2'' - 2y_2 = x^2 \left( \frac{2}{x^3} \right) - 2 \left( \frac{1}{x^2} \right) = \frac{2}{x} - \frac{2}{x} = 0.
\]
So, both \( y_1 \) and \( y_2 \) are solutions.

Finally, we see if \( y = C_1 x^2 + C_2 \frac{1}{x} \) satisfies the equation.

Noting that \( y'' = 2C_1 + 2C_2 \frac{1}{x^3} \), we have that
\[
x^2 y'' - 2y = x^2 \left( 2C_1 + 2C_2 \frac{1}{x^3} \right) - 2 \left( C_1 x^2 + C_2 \frac{1}{x} \right)
= 2C_1 x^2 - 2C_1 x^2 + 2C_2 \frac{1}{x} - 2C_2 \frac{1}{x} = 0.
\]

Thus, any linear combination of \( y_1 \) and \( y_2 \) is a solution to \( Ly = 0 \).
(b) With $Ly = yD^2y + (Dy)^2$, write $Ly = 0$ as a differential equation. Verify that

$$y_1(x) = 1 \text{ and } y_2(x) = \sqrt{x}$$

are solutions to this differential equation, for $x > 0$. If $C_1$ and $C_2$ are constants, is the linear combination $C_1y_1(x) + C_2y_2(x)$ a solution to this differential equation, for $x > 0$?

**Solution**

The differential equation $Ly = 0$ is

$$y \frac{dy}{dx}^2 + \left( \frac{dy}{dx} \right)^2 = 0.$$ 

Now, we check if $y_1$ is a solution. Noting that $y_1' = 0$ and $y_1'' = 0$, we have

$$y_1 \cdot y_1'' + (y_1')^2 = (1)(0) + (0)^2 = 0.$$ 

The function $y_1$ is a solution.

Next, we check if $y_2$ is a solution. Noting that $y_2' = \frac{1}{2\sqrt{x}}$ and $y_2'' = -\frac{1}{4x^{3/2}}$, we have

$$y_2 \cdot y_2'' + (y_2')^2 = (\sqrt{x}) \left( -\frac{1}{4x^{3/2}} \right) + \left( \frac{1}{2x^{1/2}} \right)^2 = -\frac{1}{4x} + \frac{1}{4x} = 0.$$ 

So, both $y_1$ and $y_2$ are solutions.

Finally, we see if $y = C_1 + C_2\sqrt{x}$ satisfies the equation. Noting that $y' = C_2 \frac{1}{2\sqrt{x}}$ and $y'' = -C_2 \frac{1}{4x^{3/2}}$, we have that

$$y \cdot y'' + (y')^2 = (C_1 + C_2\sqrt{x}) \left( -C_2 \frac{1}{4x^{3/2}} \right) + \left( C_2 \frac{1}{2\sqrt{x}} \right)^2$$

$$= -C_1C_2 \frac{1}{4x^{3/2}} \neq 0 \text{ in general.}$$

Thus, the linear combination $y = C_1 + C_2\sqrt{x}$ is a solution to $Ly = 0$ only when either $C_1$ or $C_2$ is equal to zero. In general (for arbitrary values of $C_1$ and $C_2$), the function $y = C_1 + C_2\sqrt{x}$ is not a solution to $Ly = 0$. 
