Written Assignment #9: Applications of Linear Equations (solutions)

1. The motion of a damped spring-mass system is called “quasi-periodic”. This means that the time it takes for the mass to complete one cycle (one upswing followed by one down-swing) is constant, but the amplitude of the motion can change with each cycle. The “quasi-period” is the length of time for the mass to complete one cycle. Suppose a mass of 400g is attached to a spring with a spring-constant of 3600 g/sec and that the observed quasi-period is 3 seconds. What is the damping constant for the spring?

Solution

The equation for the motion of the mass is

$$400x'' + cx' + 3600x = 0,$$

where the damping constant $c$ is to be determined. To identify the quasi-period for the motion, we need to solve this equation (assuming that $c^2 < 4km = 5760000$):

$$400D^2 + cD + 3600 = 0 \Rightarrow D = -\frac{c}{800} \pm \frac{\sqrt{c^2 - 5760000}}{800}.$$

It follows that the general solution is

$$x(t) = e^{-\frac{c}{800}t} \left[ C_1 \cos \left( \frac{\sqrt{5760000 - c^2}}{800} t \right) + C_2 \sin \left( \frac{\sqrt{5760000 - c^2}}{800} t \right) \right].$$

The “quasi-circular frequency” for this solution is

$$\omega = \frac{\sqrt{5760000 - c^2}}{800} \text{ radians/sec}.$$

Thus, the “quasi-period” is

$$\frac{2\pi}{\omega} = \frac{1600\pi}{\sqrt{5760000 - c^2}} \text{ sec}.$$

We want this to be 3 sec, so

$$\frac{1600\pi}{\sqrt{5760000 - c^2}} \text{ sec} = 3 \text{ sec} \Rightarrow \sqrt{5760000 - c^2} = \frac{1600\pi}{3} \Rightarrow c = \pm \sqrt{5760000 - \left( \frac{1600\pi}{3} \right)^2}.$$

Since $c$ must be non-negative, we see that

$$c = \sqrt{5760000 - \left( \frac{1600\pi}{3} \right)^2} \approx 1718.33 \text{ g/sec}.$$
2. Consider an RC circuit (a circuit with a resistor, capacitor and voltage source) with resistance of 10 ohms and a capacitor of 500 microfarads. Suppose that the circuit is hooked up to a battery, which produces a constant voltage (i.e. \( V(t) = V_0 \) is a constant). Show that the amount of charge stored in the capacitor tends to a finite limit as \( t \to \infty \), while the current in the circuit tends to 0 as \( t \to \infty \). Assuming that the initial charge on the capacitor is 0, show that regardless of what \( V_0 \) is, the time for the capacitor to reach 90\% of its limiting charge is always the same.

**Solution**

The equation for the total charge in the circuit is

\[
10Q' + 2000Q = V_0.
\]

Solving this equation we find that

\[
Q_h = C_1 e^{-200t} \quad \text{and} \quad Q_p = \frac{V_0}{2000} \Rightarrow Q(t) = C_1 e^{-200t} + \frac{V_0}{2000}.
\]

Noting that \( \lim_{t \to \infty} e^{-200t} = 0 \), we find that

\[
\lim_{t \to \infty} Q(t) = \lim_{t \to \infty} \left[ C_1 e^{-200t} + \frac{V_0}{2000} \right] = \frac{V_0}{2000}.
\]

For the current, \( I(t) = Q'(t) \), so

\[
\lim_{t \to \infty} I(t) = \lim_{t \to \infty} \left[ -200C_1 e^{-200t} \right] = 0.
\]

Imposing the initial condition \( Q(0) = 0 \), yields

\[
Q(t) = -\frac{V_0}{2000} e^{-200t} + \frac{V_0}{2000}.
\]

We want to know when \( Q(t) = .9 \left( \frac{V_0}{2000} \right) \):

\[
Q(t) = -\frac{V_0}{2000} e^{-200t} + \frac{V_0}{2000} = .9 \left( \frac{V_0}{2000} \right) \Rightarrow -e^{-200t} + 1 = .9
\]

\[
\Rightarrow -200t = \ln(.1) = -\ln(10)
\]

\[
\Rightarrow t = \frac{\ln(10)}{200}.
\]

Hence, \( t = \frac{\ln(10)}{200} \approx .0115 \text{ sec.} \) This time does not depend upon the value of \( V_0 \).