Sturm-Liouville Equations
Math 240 Extra Credit
due December 10

The approach to solving the linear partial differential equations we have encountered so far is to guess a simple form of solution (separation of variables), and then use the fact that the equation is linear to justify building complicated solutions from the simple solutions. Finding the simple solutions usually isn’t too difficult. Since we are dealing just with linear equations, we can also count on being able to combine simple solutions to form more complicated solutions. However, when we want to build a solution satisfying specific initial and boundary conditions, we will need techniques for figuring out how to build a solution with the particular values needed. For equations such as the wave equation, where the simple solutions are sines and cosines, we can use Fourier series to build solutions to specific initial and boundary value problems. The key fact that made it easy to compute Fourier series for specific functions was that

$$\int_0^\pi \sin(nx)\sin(mx)\,dx = 0 \text{ for } n \neq m.$$  

We will see that similar facts hold for a variety of functions, so that we can use techniques similar to Fourier series in many problems.

We will begin by introducing a couple of additional ideas from the general theory of linear equations (not necessarily linear differential equations). Suppose $L$ is a linear operator. Suppose we have a non-zero function $f$ with $Lf = \lambda f$ where $\lambda$ is a constant (saying the function is non-zero means that it is not always 0, there may be specific values of $x$ for which $f(x) = 0$). We call such a function $f$ an eigenfunction of the operator $L$ and we call $\lambda$ an eigenvalue (these are also sometimes called characteristic functions and characteristic values).

1. Show that $\sin(nx)$ is an eigenfunction for the operator $L = \frac{d^2}{dx^2}$ with eigenvalue $-n^2$.

Next, we introduce an analogue of the dot product of vectors, which we will call an inner product. Recall from Calculus III that the dot product of two vectors is computed by multiplying the corresponding coordinates of each vector and then summing up the products, so $(x, y, z) \cdot (r, s, t) = rx + sy + tz$. We define the inner product of two functions $f(x)$ and $g(x)$ as

$$\langle f, g \rangle = \int_0^\pi f(x)g(x)\,dx.$$  

You should note that just as in dot products of vectors, we’ve multiplied corresponding terms of the functions and then summed up the products in an integral. You can easily check that this inner product has the same properties as the dot product of vectors such as $\langle f, g \rangle = \langle g, f \rangle$, and in particular, if $a$ is a constant, then $\langle af, g \rangle = \langle f, ag \rangle = a \langle f, g \rangle$. Next, we call a linear operator $L$ self-adjoint with respect to an inner product if $\langle Lf, g \rangle = \langle f, Lg \rangle$. 
2. Suppose we restrict our work to functions \( f(x) \) and \( g(x) \) such that 
\[ f(0) = g(0) = 0 \text{ and } f(\pi) = g(\pi) = 0. \]
Show that \( L = \frac{d^2}{dx^2} \) is self-adjoint with respect to the inner product we just defined. Hint: integrate by parts twice.

In terms of our inner product, the key result for sines that made Fourier series easy to work with was that \( \langle \sin(nx), \sin(mx) \rangle = 0 \) when \( n \neq m \). But this will turn out to be a general property of eigenfunctions of self-adjoint operators.

3. Suppose \( f \) and \( g \) are eigenfunctions of a self-adjoint operator \( L \), with eigenvalues \( a \) and \( b \) respectively. Show that if \( a \neq b \), then \( \langle f, g \rangle = 0 \). Hint: show 
\[ a \langle f, g \rangle = b \langle f, g \rangle, \]
and then conclude \( \langle f, g \rangle = 0 \) if \( a \neq b \).

The process of separating variables tends to lead to simple solutions built out of eigenfunctions of differential operators (since you end up setting things equal to a constant \( \lambda \)). So if the differential operators are self-adjoint, you will be able to build a series out of the simple solutions using the same tricks as for the Fourier series (actually, there is still some work involved in showing you will get enough eigenfunctions to build all the complicated solutions you want, but I need to leave something for you to learn in later courses). This leads to the question of what sorts of differential operators will be self-adjoint. An important class of such operators is the Sturm-Liouville operators.

4. Show \( L = a(x) \frac{d^2}{dx^2} + a'(x) \frac{d}{dx} + b(x) \) is self-adjoint with respect to the inner product we defined above if we restrict our work to functions \( f(x) \) and \( g(x) \) such that 
\[ f(0) = g(0) = 0 \text{ and } f(\pi) = g(\pi) = 0. \]
Hint: use integration by parts twice. It may help to rewrite 
\[ Lf = \frac{d}{dx} \left( a(x) \frac{df}{dx} \right) + b(x) f(x). \]

Note that there are many ways to develop inner products. We can work over intervals other than \([0, \pi]\). We can also introduce a weight function \( w(x) > 0 \), and define
\[ \langle f, g \rangle = \int_{\alpha}^{\beta} f(x)g(x)w(x)dx. \] This would correspond to weighting the contributions of the different components of the dot product differently when we sum them up. By such manipulations, you can often find an appropriate inner product for which your differential operator is self-adjoint, so you can use techniques similar to the Fourier series to solve your problems, just with a different inner product. But we will leave this topic for a later course.