1. Solve the initial value problem,
\[ x'' + 6x' + 10x = \delta(t), \quad x(0) = 0, \quad x'(0) = 0. \]

**Constant Coefficient Initial Value Problem \(
\mathbf{\rightarrow \text{Laplace Transform}}
\)**

\[ \mathcal{L}\{x'' + 6x' + 10x\} = \mathcal{L}\{\delta(t)\} \]

\[ s^2 \mathcal{L}\{x\} - sx(0) - x'(0) + 6s \mathcal{L}\{x\} - 6x(0) + 10 \mathcal{L}\{x\} = 1 \]

\[ (s^2 + 6s + 10) \mathcal{L}\{x\} = 1 \]

\[ \mathcal{L}\{x\} = \frac{1}{s^2 + 6s + 10} = \frac{1}{s^2 + 6s + 9 + 1} \]

\[ = \frac{1}{(s + 3)^2 + 1^2} \]

\[ x = e^{-3t} \sin(t) u(t) \]
2. Find all solutions to \( x^2 y'' + 5x y' + 3y = 0 \).

**Euler Equation**

\[
\begin{align*}
y &= x^r \\
y' &= r x^{r-1} \\
y'' &= r(r-1)x^{r-2}
\end{align*}
\]

\[
x^2 y'' + 5x y' + 3y = r(r-1)x^r + 5rx^r + 3x^r = 0
\]

\[
(r^2 - r + 5r + 3)x^r = 0
\]

\[
(r^2 + 4r + 3)x^r = 0
\]

\[
(r + 3)(r + 1) = 0
\]

\[
r = -3, -1
\]

\[
y = C_1 x^{-3} + C_2 x^{-1}
\]
3. Solve the initial value problem
\[ y'' + xy' + 2y = 0, \quad y(0) = 1, \quad y'(0) = 1. \]

Use \( x_0 = 0 \)

**Variable Coeff \rightarrow Series Soln.**

**Step 1**
\[
y = \sum_{n=0}^{\infty} a_n x^n \quad \text{and} \quad y' = \sum_{n=0}^{\infty} n a_n x^{n-1} \quad \text{and} \quad xy' = \sum_{n=1}^{\infty} n a_n x^n
\]

**Step 2**
\[
y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}
\]

**Step 3**
\[
y'' = \sum_{n=2}^{\infty} (j+2)(j+1)a_{j+2} x^j
\]

**Step 4**
\[
\sum_{m=0}^{\infty} (m+2)(m+1)a_{m+2} x^m + \sum_{m=0}^{\infty} ma_m x^m + \sum_{m=0}^{\infty} 2a_m x^m = 0
\]

**Step 5**
\[
(2a_2 + 2a_0) + \sum_{m=1}^{\infty} [(m+2)(m+1)a_{m+2} + (m+2)a_m] x^m = 0
\]

**Step 6**
\[
a_0 = y(0) = 1 \quad a_4 = -\frac{a_2}{3} = \frac{1}{3}
\]
\[
a_1 = y'(0) = 1 \quad a_5 = -\frac{a_3}{4} = \frac{1}{8}
\]
\[
a_2 = -a_0 = -1 \quad a_6 = -\frac{a_4}{5} = -\frac{1}{15}
\]

\[
ap_{m=1}: \quad a_3 = -\frac{a_1}{2} = -\frac{1}{2}
\]
\[
ap_{m=0}: \quad a_3 = -\frac{a_1}{2} = -\frac{1}{2}
\]

\[
y(x) = 1 + x - x^2 - \frac{1}{2} x^3 + \frac{1}{3} x^4 + \frac{1}{4} x^5 + \frac{1}{5} x^6 + \cdots
\]
4. Solve the initial value problem
\[
\frac{dx}{dt} = x + 2y, \quad x(0) = 1,
\]
\[
\frac{dy}{dt} = -x + 4y, \quad y(0) = 0.
\]

Constant Coefficient System \rightarrow Laplace Transform
(\text{Let } \mathbf{X} = L\{x\}, \quad \mathbf{Y} = L\{y\})

\[
s\mathbf{X} - x(0) = \mathbf{X} + 2\mathbf{Y}
\]
\[
s\mathbf{Y} - y(0) = -\mathbf{X} + 4\mathbf{Y}
\]

\[
(s-1)\mathbf{X} - 2\mathbf{Y} = 1
\]
\[
\mathbf{X} + (s-4)\mathbf{Y} = 0
\]

\[
\mathbf{X} = -(s-4)\mathbf{Y} = \frac{s-4}{s^2 - 5s + 6}
\]

\[
\frac{A}{s-3} + \frac{B}{s-2} = \frac{s-4}{(s-3)(s-2)}
\]
\[
A(s-2) + B(s-3) = s-4
\]
\[
A = -1
\]
\[
B = 2
\]

\[
\mathbf{X} = \frac{-1}{s-3} + \frac{2}{s-2}
\]

\[
\mathbf{X} = 2e^{2t} - e^{3t}
\]
\[
y = e^{2t} - e^{3t}
\]
5. (a) Find the Laplace transform of $e^{-2t} \cos(3t)$.

\[
\frac{s+2}{(s+2)^2 + 3^2} = \frac{s+2}{s^2 + 4s + 13}
\]

(b) Find the inverse Laplace transform of $\frac{2s + 3}{s^3 + 4s^2 + 13s}$.

\[
\frac{2s + 3}{s(s^2 + 4s + 13)} = \frac{2s + 3}{s[2(s+2)^2 + 3^2]} = A \frac{1}{s} + B \frac{s+2}{(s+2)^2 + 3^2} + C \frac{3}{(s+2)^2 + 3^2}
\]

\[
2s + 3 = A[(s+2)^2 + 3^2] + B(s+2)s + 3Cs
\]

\[
2s + 3 = As^2 + 4As + 13A + Bs^2 + 2Bs + 3Cs
\]

\[
5^2: \quad 0 = A + B \quad B = -\frac{3}{13}
\]

\[
5: \quad 2 = 4A + 2B + 3C \quad C = \frac{2A}{3} = \frac{2}{3} + \frac{2}{3} + 3C
\]

\[
3 = 13A \quad A = \frac{3}{13} \quad 2/3 = 3C
\]

\[
\frac{3}{13} \cdot 1 - \frac{3}{13} e^{-2t} \cos(3t) + \frac{2}{39} e^{-2t} \sin(3t)
\]
6. Find a lower bound for the radius of convergence of the series solution for \((x^3 + 1)y'' + 3xy' + (2x - 5)y = 0\) about \(x_0 = 3\) .

Singular Points \(x^3 + 1 = 0\) \(<\) Note: 3 roots since cubic

\[
x = -1
\]

This gives us one root, then factor that root out to get a quadrate

\[
x + 1 \\
\frac{x^3 + 1}{x + 1} \\
\frac{x^3 + x^2}{x + 1} \\
\frac{-x^2 - x}{x + 1} \\
\frac{x + 1}{x + 1}
\]

So \(x^3 + 1 = (x+1)(x^2-x+1)\)

Roots \(-1, \frac{1}{2} \pm \frac{\sqrt{3}}{2}i\) \(<\) use quadratic formula

\[
-\frac{1}{2} + \frac{\sqrt{3}}{2}i
\]

\[
\sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \sqrt{\frac{25}{4} + \frac{3}{4}}
\]

\[
= \sqrt{\frac{28}{4}}
\]

\[
= \sqrt{7}
\]

Closest singular point \(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\)

So radius of convergence \(> \sqrt{7}\)
7. Match the following graphs of solutions to initial value problems on the right with the graphs of the poles of their Laplace transforms on the left.

- Complex roots with negative real part correspond to decaying oscillations.
- Negative real roots correspond to exponential decay.
- Roots on imaginary axis correspond to steady state.
8. Write a paragraph about series solutions to differential equations. Your paragraph may include equations and may refer to graphs you draw. In grading the paragraph I will consider both content and clarity and will look for the following specific elements.

(a) Why would we want to use a series solution to a differential equation?

(b) What is meant by the term "radius of convergence" for a series solution? Why do we use the word radius?

(c) If the series solution to a differential equation diverges in a region, does it follow that every representation of the solution must be undefined in that region? Can you give an example?

While a linear differential equation with nice (but non-constant) coefficients is guaranteed to have a solution, it will usually be the case that the solution function can't be expressed in terms of our familiar functions, such as the trigonometric, exponential, and rational functions. In such cases we will need to define new types of functions, and we can do that via power series, which lead to our series solutions. Now the power series is just one possible representation of the solution function. A power series of the form \( \sum_{n=0}^{\infty} a_n (x-x_0)^n \) will always converge in a region of the form \( |x-x_0| < R \), where \( R \) is called the radius of convergence. \( R \) is called a "radius" because \( |x-x_0| < R \) defines a disk in the complex plane with radius \( R \). However, just because the series solution converges in such a region, we can't conclude there isn't another representation of the solution function which is defined in a larger region. For example, during lab we saw that the Taylor series expansion for \( \frac{1}{1-x} \) only converged for \( |x| < 1 \), while the function is actually defined for all real \( x \).