Fourier Series and the Wave Equation – Part 2

There are two big ideas in our work this week. The first is the use of linearity to break complicated problems into simple pieces. The second is the use of the symmetries of the sine function to help analyze complicated functions as sums of simple functions. On Monday, we used the first idea to show how we could take the very simple solutions of the wave equation found using separation of variables and put them together to build more complicated solutions. We used the second idea when we used the orthogonality of the functions $\sin(nx)$ over the range from 0 to $\pi$ to write a general function as a Fourier series. We will develop both these general themes further today.

On Monday, we considered the problem

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2},$$
$$u(0,t) = u(\pi,t) = 0,$$
$$u(x,0) = f(x),$$
$$\frac{\partial u}{\partial t}(x,0) = 0.$$  \hspace{1cm} (1)

In this problem, we had a vibrating string whose endpoints were fixed at 0 and whose initial position was given by the function $f(x)$, starting with initial velocity 0. This last condition is not particularly reasonable from a physical standpoint. If you pluck a guitar string and let it vibrate, the plucking not only changes the initial position but also imparts an initial velocity. So we should instead look at a problem of the form

$$\frac{\partial^2 w}{\partial t^2} = \frac{\partial^2 w}{\partial x^2},$$
$$w(0,t) = w(\pi,t) = 0,$$
$$w(x,0) = f(x),$$
$$\frac{\partial w}{\partial t}(x,0) = g(x).$$  \hspace{1cm} (2)

where $g(x)$ denotes the initial velocity. The problem is that we used the fact that the initial velocity was 0 as part of our simplifying our large collection of solutions generated by separation of variables down to a small set which we could use to build a Fourier series. The key is to apply our basic idea of linearity again. We will split problem (2) into two parts and solve each part separately. This should remind you of how we solved linear equations in chapter 2 by looking for the homogeneous and particular solutions separately. Both approaches are based on the key advantage of linear equations, that you can split large problems into small problems, solve the small problems separately, and then add the solutions to the small problems together to solve the large problem.

Since we already know how to solve problem (1), where the initial position is given and the initial velocity is 0, what we have to do to finish solving the full problem (2) is to work with a given velocity and a 0 initial position. This leads to the problem
We solve this problem in a manner almost identical to our solution to problem (1) on Monday. From the differential equation \( \frac{\partial^2 v}{\partial t^2} = \frac{\partial^2 v}{\partial x^2} \) and the boundary conditions \( v(0,t) = v(\pi,t) = 0 \) we conclude that \( v_n(x,t) = (a_n \cos(nt) + b_n \sin(nt))\sin(nx) \). Then we use the requirement that \( v(x,0) = 0 \) for all \( x \) to conclude \( a_n = 0 \) and hence that \( v_n(x,t) = b_n \sin(nt)\sin(nx) \). We now form a more general solution by adding these simple solutions to get \( v(x,t) = \sum_{n=1}^\infty b_n \sin(nt)\sin(nx) \). Next we compute \( \frac{\partial v}{\partial t}(x,0) = \sum_{n=1}^\infty nb_n \sin(nx) = g(x) \) to see that all we need to do is find another Fourier series expansion. For example, if \( g(x) = \begin{cases} x, & 0 \leq x \leq \frac{\pi}{2} \\ \pi - x, & \frac{\pi}{2} \leq x \leq \pi \end{cases} \), then from our work Monday we know that \( g(x) = \sum_{n=0}^{\infty} (-1)^n \frac{4}{\pi(2n+1)^2} \sin((2n+1)x) \), so

\[
\frac{\partial v}{\partial t}(x,0) = \sum_{n=1}^\infty nb_n \sin(nx) = g(x).
\]

Monday we know that \( g(x) = \sum_{n=0}^{\infty} (-1)^n \frac{4}{\pi(2n+1)^2} \sin((2n+1)x) \), so

\[
b_n = \begin{cases} 0, & \text{if } n \text{ even;} \\ (-1)^{\frac{n-1}{2}} \frac{4}{\pi n^2}, & \text{if } n \text{ odd.} \end{cases}
\]

Note the fact that we had \( nb_n \) as the coefficient of our series for \( g(x) \) led to the extra power of \( n \) in the denominator of \( b_n \), compared to our result from Monday. Using the fact that \( 2n+1 \) runs through all the odd integers, we can then write our final solution to problem (2) as

\[
v(x,t) = \sum_{n=0}^{\infty} (-1)^n \frac{4}{\pi(2n+1)^2} \sin((2n+1)x) \sin((2n+1)t).
\]

Finally, we put our solutions to (1) and (3) together to solve the full problem (2). Let \( w(x,t) = u(x,t) + v(x,t) \) where \( u \) and \( v \) are the solutions we found to problems (1) and (3) respectively. Then
\[
\frac{\partial^2 w}{\partial t^2} = \frac{\partial^2 (u + v)}{\partial t^2} = \frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 v}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x^2} = \frac{\partial^2 (u + v)}{\partial x^2},
\]

\[
w(0,t) = u(0,t) + v(0,t) = 0 + 0 = 0,
\]

\[
w(\pi, t) = u(\pi, t) + v(\pi, t) = 0 + 0 = 0,
\]

\[
w(x, 0) = u(x, 0) + v(x, 0) = f(x) + 0 = f(x),
\]

\[
\frac{\partial w}{\partial t}(x, 0) = \frac{\partial (u + v)}{\partial t}(x, 0) = \frac{\partial u}{\partial t}(x, 0) + \frac{\partial v}{\partial t}(x, 0) = 0 + g(x) = g(x),
\]

where we have used the fact that both our equation and our conditions are linear to show that the solution to the full problem (2) with \( f(x) = g(x) = \)

\[
\begin{cases}
  x, & 0 \leq x \leq \frac{\pi}{2} \\
  \pi - x, & \frac{\pi}{2} \leq x \leq \pi
\end{cases}
\]

is

\[
w(x, t) = \sum_{n=0}^{\infty} (-1)^n \frac{4}{\pi(2n + 1)^2} \left[ \frac{1}{n} \sin \left( (2n+1)t \right) + \cos \left( (2n+1)t \right) \right] \sin \left( (2n+1)x \right).
\]

Note that the condition that the initial position and initial velocity are equal is not quite as unlikely as you might think. After all, the points that have been pulled farthest from equilibrium when the string is plucked are going to be moving faster than the parts that haven’t been moved as far from equilibrium. On the homework you have a slightly more general problem, in which the endpoints are fixed at points \( a \) and \( b \) rather than at 0. This just adds one more simple problem to be solved, and then you add things up just as we did here to find the full solution.

The other big idea we’ve worked with is using the symmetry properties of the sine function to simplify the process of writing a complicated function as a sum of sine terms. You should have further developed a sense of how these symmetry properties work during the lab on Tuesday. In particular, you should have noticed that if the function \( f(x) \) is symmetric about \( x = \pi/2 \), then the even coefficients of the Fourier series are all 0, while if \( f(x) \) is anti-symmetric about \( \pi/2 \) then the odd coefficients are 0. This, and the other facts we need about sine functions, can be proved with the aid of some trig identities.

\[
\sin(\theta)\sin(\varphi) = \frac{1}{2}(\cos(\theta - \varphi) - \cos(\theta + \varphi))
\]

\[
\sin^2(\theta) = \frac{1}{2} - \frac{1}{2}\cos(2\theta)
\]

\[
\sin(\theta) = \sin(2\pi n + \theta)
\]

\[
\sin\left( \frac{n\pi}{2} + \theta \right) = -\sin\left( \frac{n\pi}{2} - \theta \right), \quad \text{if } n \text{ even}
\]

\[
\sin\left( \frac{n\pi}{2} + \theta \right) = \sin\left( \frac{n\pi}{2} - \theta \right), \quad \text{if } n \text{ odd}
\]
For example, suppose we want to show that if $f(x)$ is anti-symmetric about $\frac{\pi}{2}$, i.e. $f\left(\frac{\pi}{2} - x\right) = -f\left(\frac{\pi}{2} + x\right)$, then all the odd coefficients in the Fourier series for $f(x)$ are 0. I will write out the steps and then explain them at the end.

$$a_m = \frac{2}{\pi} \int_{0}^{\pi} f(x) \sin(mx) \, dx$$

$$= \frac{2}{\pi} \left[ \int_{0}^{\pi/2} f(x) \sin(mx) \, dx + \int_{0}^{\pi/2} f(x) \sin(mx) \, dx \right]$$

$x = \frac{\pi}{2} - u$, $x = \frac{\pi}{2} + u$

$dx = -du$, $dx = du$

$$= \frac{2}{\pi} \left[ \int_{0}^{\pi/2} f\left(\frac{\pi}{2} - u\right) \sin\left(\frac{m\pi}{2} - mu\right)(-du) + \int_{0}^{\pi/2} f\left(\frac{\pi}{2} + u\right) \sin\left(\frac{m\pi}{2} + mu\right) du \right]$$

$$= \frac{2}{\pi} \left[ \int_{0}^{\pi/2} f\left(\frac{\pi}{2} - u\right) \sin\left(\frac{m\pi}{2} - mu\right) du - \int_{0}^{\pi/2} f\left(\frac{\pi}{2} + u\right) \sin\left(\frac{m\pi}{2} + mu\right) du \right]$$

$$= \frac{2}{\pi} \int_{0}^{\pi/2} f\left(\frac{\pi}{2} - u\right) \left[ \sin\left(\frac{m\pi}{2} - mu\right) - \sin\left(\frac{m\pi}{2} + mu\right) \right] du$$

$$= \frac{2}{\pi} \int_{0}^{\pi/2} f\left(\frac{\pi}{2} - u\right) \left[ \sin\left(\frac{m\pi}{2} - mu\right) - \sin\left(\frac{m\pi}{2} + mu\right) \right] du$$

$$= \frac{2}{\pi} \int_{0}^{\pi/2} f\left(\frac{\pi}{2} - u\right) \left[ \sin\left(\frac{m\pi}{2} - mu\right) - \sin\left(\frac{m\pi}{2} + mu\right) \right] du$$

$$= 0.$$

We start with the definition of the Fourier coefficients $a_m = \frac{2}{\pi} \int_{0}^{\pi} f(x) \sin(mx) \, dx$ and split the integral into two pieces, one from 0 to $\frac{\pi}{2}$ and one from $\frac{\pi}{2}$ to $\pi$ in the second line. We then make the indicated change of variables in the third line, where the $-u$ sign on the $du$ switches the upper and lower limits of the first integral in the fourth equation. Then we use the anti-symmetry condition we are given for $f(x)$ in the fifth equation. We then write the difference of the integrals as the integral of the difference, where we factor out the common term $f\left(\frac{\pi}{2} - u\right)$. Finally, since $m$ is odd, the last trig identity from the previous page shows the difference in the brackets is 0, and hence the whole integral is 0.
There are a lot of tricks like this you can develop. It shouldn’t be surprising that
\[ \int_0^\pi \sin(nx) \sin(mx) \, dx = 0 \quad \text{for} \quad n \neq m \] (which is one of the key facts we used in finding the Fourier series), since \( \sin(nx) \) and \( \sin(mx) \) will have different symmetries. You are asked to show several results like this on the homework. You are also asked to find the Fourier series for one particular function and then solve a problem similar to problem (1) where you use that function for the initial position, which isn’t hard once you’ve got the Fourier series.

While we have concentrated on applications of Fourier series to differential equations (since this is a differential equations course after all), it should be noted they have applications in many different areas. Fourier series, and especially the discrete analogue, the Fast Fourier Transform, play a crucial role in signal analysis and data compression. Furthermore, many of the tricks we developed this week will work for other sets of functions that will be better adapted to different problems. The extra credit assignment takes you through a standard approach to building such series for Sturm-Liouville problems, which use some of the deeper applications of linearity to see how the orthogonality condition we use for Fourier series will work in a wide variety of functions that arise in variable coefficient problems. We will look more closely at a specific example, the wave equation for a circular drum, on Monday, and show how we can explain mathematically the tonal differences between drums (vibrating circles) and violins (vibrating strings).