Review for Exam 3 - Fall 2001

Fall 1987

1. Find the following Laplace transforms and inverse Laplace transforms.
   a) \( \mathcal{L}\{\cos(3t)\} \)
   b) \( \mathcal{L}^{-1}\left\{ \frac{1}{s^2+6s+18} \right\} \)

2. Solve the initial value problem:
   \( x'' + 14x' + 40x = 0 \)
   \( x(0) = 2, x'(0) = 1 \)

3. Solve the initial value problem:
   \( x'' + 6x' + 18x = \delta(t) \)
   \( x(0) = 0, x'(0) = 0 \)

4. Solve the initial value problem:
   \( x'' + 10x' + 16x = f(t) \)
   \( x(0) = 0, x'(0) = 0 \)

5. Solve the initial value problem:
   \( y'' + 2xy' + (x - 1)y = 0 \)
   \( y(0) = 1, y'(0) = 1 \)

6. Find a lower bound for the radius of convergence of the series solution about
   \( x_0 = 0 \) to :
   \( (x^2 + 4x + 5)y'' + (x - 3)y' + (2x + 1)y = 0 \)

7. Explain what knowing the radius of convergence in problem 6 tells you about
   the series solution.

8. Match the functions and their power series expansions:
   a) \( \cos(x) \quad \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \)
   b) \( \sinh(x) \quad \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} \)
   c) \( \frac{\sin(x)}{x} \quad \sum_{n=0}^{\infty} x^n \)
1. Solve the initial value problem:
\[ y'' + y' + 4y = \delta(t), \quad y(0) = 0, \quad y'(0) = 0. \]

2. Find the general solution to \( y'' + (4x + 2)y' + 8xy = 0. \)

3. Solve the initial value problem:
\[ y'' + 2y' + 5y = f(t), \quad y(0) = 1, \quad y'(0) = 0. \]

4. Find a lower bound for the radius of convergence of the power series solution to \((x^2 + 4x + 20)y'' + (3x + 2)y' + 4y = 0, \quad y(0) = 1, \quad y'(0) = 0, \) about the point \( x_0 = 0. \)

5. Suppose \( y(x) = \sum_{n=0}^\infty a_n x^n \) solves: \( y'' + (x + 1)y' + y = 0, \quad y(0) = 1, \quad y'(0) = 0. \) Approximate \( y(1.2) \) using the eighth-order Taylor polynomial approximation to the solution. You may use as given that the coefficients of the series solution satisfy the equations:
\[
\begin{align*}
a_0 + a_1 + 2a_2 &= 0 \\
(n + 1)(n + 2)a_{n+2} + (n + 1)a_{n+1} + (n + 1)a_n &= 0 \quad (\text{for } n \geq 1)
\end{align*}
\]
Your answer should be given to the nearest 0.001.

6. Suppose \( y(x) = \sum_{n=0}^\infty a_n x^n. \) Prove that \( y(0) = a_0 \) and \( y'(0) = a_1, \) but that \( y''(0) \) need not equal \( a_2. \) (Hint: differentiate the series term-by-term and then substitute \( x = 0 \) and evaluate the resulting sum, which will consist of a single non-zero term).

8. Suppose the series solution to a differential equation about \( x_0 = 0 \) has a radius of convergence of 1. Does that mean the differential equation has no solution for \(| x | \geq 1? \) Justify your answer.
Spring 2000

1. Solve the initial value problem:
   \[ y'' + 4y' + 3y = \delta(t), \quad y(0) = 0, \quad y'(0) = 0. \]

2. Find the general solution to \( x^2y'' + 7xy' + 8y = 0. \)

3. Solve the initial value problem:
   \[ y'' + 4y' + 5y = f(t), \quad y(0) = 0, \quad y'(0) = 0. \]

4. Solve the initial value problem \( y'' + xy' + 3y = 0, \quad y(0) = 1, \quad y'(0) = 1. \)

5. Find a lower bound for the radius of convergence of the series solution:
   \[ y(x) = \sum_{n=0}^{\infty} a_n x^n \]
   to
   \[ (x^3 + x^2 - 4x - 24)y'' + (7x - 9)y' + (x^4 + 1)y = 0, \]
   \[ y(0) = 1, \quad y'(0) = 2. \]
   Hint: \( x_0 = 3 \) is a singular point.

6. Suppose \( y_1(x) \) and \( y_2(x) \) are two series solutions about \( x_0 = 0 \) to a second order linear differential equation \( p(x)y'' + q(x)y' + r(x)y = 0, \) (with \( p(0) \neq 0 \)), produced by the paradigm in section 4.3. Prove that \( y_1(x) \) and \( y_2(x) \) are linearly independent. Hint: Show that if \( y(x) = \sum_{n=0}^{\infty} a_n x^n \), then \( y(0) = a_0 \) and \( y'(0) = a_1 \). From this and the rules of the paradigm, you can determine \( y_1(0), y_1'(0), y_2(0), y_2'(0) \) without having to find the series solution. Then compute the Wronskian of \( y_1(x) \) and \( y_2(x) \) and show it is non-zero at \( x = 0. \)

8. Explain the limitations of a series solution to a differential equation, in terms of actually computing the values of the solution at a specified point. For full credit, use complete sentences in your explanation and include at least two limitations in your discussion. For each limitation, include at least one sentence of explanation as well as an example from the labs.
Fall 2000

1. Solve: \( x'' + 16x = 3\delta(t - 1) \), \( x(0) = 0 \), \( x'(0) = 0 \).

2. Refer to your solution in problem 1 to the initial value problem \( x'' + 16x = 3\delta(t - 1) \), \( x(0) = 0 \), \( x'(0) = 0 \). If we interpret \( x(t) \) as the displacement in meters of a mass suspended on an ideal spring, and all of the coefficients as parameters of the system in kms units, this models an elastic collision at time \( t = 1 \) seconds after some chosen time \( t = 0 \).
   a) Find the energy of the system for \( t < 1 \). b) Find the energy of the system for \( t > 1 \) (If you don’t get a constant, check your work here and in problem 1). c) How much energy was transferred to the mass by elastic collision?

3. A 1.0 kg mass is attached to an ideal (undamped) spring, causing it to stretch 98 cm. If the mass is subjected to a force of \( 2.0\cos(3.0t) \) newtons, find the equation of the resulting motion (Assume the mass is instantaneously at rest and equilibrium at \( t = 0 \)).

4. a) Find the inverse Laplace transform of \( \frac{2s - 6}{s^2 + 4s + 40} \). b) Find the Laplace transform of \( 3\cos(4t) - 2\sin(4t) \).

5. Use Laplace transforms to solve: \( x'' - 2x' + x = te^t \), \( x(0) = 1 \), \( x'(0) = 0 \)

6. Use Laplace transforms to solve: \( x'' + x = f(t) \), \( x(0) = 0 \), \( x'(0) = 0 \)

7. a) Find the Taylor series of \( f(x) = \frac{x}{1 + 4x} \) at \( x_0 = 0 \). b) Find the radius of convergence for the series in part a).

8. Solve the system: \( x' = 2x + y \), \( x(0) = 1 \), \( y' = -4x - 3y \), \( y(0) = 0 \)
Spring 2001

1. a) Solve: $x'' + 3x = 3\delta(t - 1), \quad x(0) = 0, \quad x'(0) = 0$

b) If we interpret $x(t)$ as the displacement in meters of a mass suspended on an ideal spring, and all of the coefficients as parameters of the system in kms units, this models an elastic collision at time $t = 1$ seconds after some chosen time $t = 0$. How much energy did the elastic collision transfer to the mass?

2. Use Laplace transforms to solve: $x'' + 9x = \sin(t) \quad x(0) = 1, x'(0) = 0$

3. Solve by power series at $x_0 = 0$. As your answer give the recurrence relation for the later coefficients and the four non-zero terms of the series (or all the terms if there are fewer than four non-zero terms): $y'' + xy' + x^2y = 0, \quad y(0) = 1, \quad y'(0) = 0$

4. Use the Laplace transform to solve: $x' = x + y \quad x(0) = 0, \quad y' = x - y \quad y(0) = 0$

5. Suppose we have solved $(x^2 + 3)y'' - 4y = 0$ by power series. Give a lower bound on the radius of convergence. a) If we worked about $x_0 = 0$. b) If we worked about $x_0 = 1$.

6. Solve: $x^2y'' - 2xy' - 4y = 0, \quad y(1) = 1, \quad y'(1) = 1$

7. Use Laplace transforms to solve: $x'' + 2x' + x = 4e^{-t} \quad x(0) = 0, \quad x'(0) = 1$

8. Solve by power series about $x_0 = 0$: $y'' - x^2y = \frac{1}{1+x}, \quad y(0) = 0, \quad y'(0) = 0$

Hint: $\frac{1}{1+x} = \sum_{n=0}^{\infty}(-1)^nx^n$. 