Extra Credit 3 – Simpson’s Rule
Due in Recitation, Wednesday, 3/9

The note cards for section 7.1 frequently included questions about Simpson’s rule, which apparently wasn’t covered in Calculus I last fall. Despite all the techniques we have learned, there are many integrals that we won’t be able to evaluate. In these cases, it may be necessary to find a numerical approximation to the integral, since we won’t be able to find an exact answer. Simpson’s rule is a simple method of approximating the value of a definite integral. The fundamental idea is based on remembering that the definite integral, \( \int_{a}^{b} f(x) \, dx \), is the area under the curve \( y = f(x) \) between \( x=a \) and \( x=b \). We approximate this area by picking a simple curve (which we can integrate) to approximate \( y = f(x) \) and then finding the area under the simple curve. In particular, we approximate the curve \( y = f(x) \) by a series of sections of parabolas. We then find the areas under the parabolas, which is easy since these are just quadratic polynomials. This becomes our approximation to the integral of \( f(x) \). This is spelled out in more detail in section 4.6 (remember that if you have the paperback version of the text which doesn’t have chapter 4, you can get the full hardback version from the reserve desk in either Hale or Fiedler libraries). In particular, the section will give a simple formula to use so you don’t have to work with the parabolas directly but can just evaluate the function \( f(x) \) at selected points and plug into the formula. There is also a formula that will let you determine the accuracy of the approximation, provided you can determine the maximum value of the fourth derivative, \( |f^{(4)}(x)| \) over the region of integration from \( x=a \) to \( x=b \).

For this extra credit you should read section 4.6 and then answer the following questions. The first two questions are very similar to questions in the text and should be fairly straightforward. You may use a calculator or spreadsheet to carry out the computations. The questions after that get a little more complicated.

1. Approximate the following integrals using Simpson’s rule with \( n = 4 \) and \( n = 8 \) points.
   
   a. \[ \int_{1}^{1.1} \sin(x^3) \, dx \]

   b. \[ \int_{0}^{4} \frac{4}{1 + x^2} \, dx \] (note that the exact value of this integral is \( \pi \)).
2. Use the error formula in Theorem 4.19 (p. 304) to answer the following two questions.

   a. Give a bound for the error in using Simpson’s rule to approximate
      \[ \ln(2) = \int_1^2 \frac{dx}{x} \] with \( n = 6 \) points.

   b. How many points would be needed to have an approximation of \( \ln(2) \) guaranteed (by Theorem 4.19) to be within \( 10^{-6} \)?

3. According to the error estimate in Theorem 4.19, the error will be 0 as long as the fourth derivative is 0 at all points, hence Simpson’s rule will provide exact answers for cubic polynomials (including polynomials of smaller degree as well).

   a. Evaluate \( \int_{-1}^{1} a_0 x^3 + a_2 x^2 + a_1 x + a_0 \, dx \) both using Simpson’s rule with \( n = 2 \) and exactly using the anti-derivative. Show the two values are equal.

   b. Simpson’s rule should work perfectly for quadratic polynomials, since their graphs are parabolas, so approximating the graph by a parabola should give an exact answer. Can you give a short explanation for why adding the \( x^3 \) term doesn’t cause problems in this integral? *Hint:* Simpson’s rule is symmetric about the middle point and \( x^3 \) is an odd function. While I set this up centered at \( x=0 \), it is possible to show the same sorts of symmetry ideas will work in general to explain why cubic polynomials are perfectly approximated by an integration technique based on parabolas.

4. The mean value of a function over the region from \( x=a \) to \( x=b \) is
   \[ \frac{1}{b-a} \int_a^b f(x) \, dx. \]

   Show that Simpson’s rule says that we can approximate the mean value of a function over the region from \( x = x_0 \) to \( x = x_n \) by a weighted average of the value of the function at evenly spaced points,

   \[ f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 4f(x_{n-2}) + 2f(x_{n-1}) + f(x_n) \]

   \[ \frac{1 + 4 + 2 + 4 + \cdots + 4 + 2 + 1}{1 + 4 + 2 + 4 + \cdots + 4 + 2 + 1} \]

   This seems rather an odd way to find the average of the function. While it would seem better to weight the different values equally, that would lead to the trapezoidal rule (also discussed in section 4.6) which turns out to be less accurate than Simpson’s rule in general. In fact, in a later extra credit assignment we will find that you usually can do (much) better by using irrational weights and non-uniform points. This will lead to Gaussian integration, which is the basis for the numerical integration routine in TI calculators.