Math 221 – Spring 2005
Practice Exam 3

75 minutes. Closed book. You are allowed a calculator and one 8.5” x 11” sheet of handwritten notes. Not all problems on Exam 3 will have a matching problem on this practice test, but this practice test should give you a sense of what the exam will be like as you study. You will need to show all your work to receive full credit.

1. Evaluate \( \int_{-2}^{4} \frac{dx}{x^2 + 6x + 5} \). You need to give the exact answer, not a decimal approximation.

This is an improper integral, since it has a singularity at \( x = -1 \). So we rewrite it as a pair of limits.

\[
\lim_{N \to -1} \int_{-2}^{N} \frac{dx}{x^2 + 6x + 5} + \lim_{M \to -1} \int_{M}^{4} \frac{dx}{x^2 + 6x + 5}
\]

Now we evaluate the integrals using partial fractions.

\[
\frac{1}{x^2 + 6x + 5} = \frac{A}{(x + 1)} + \frac{B}{(x + 5)}
\]

\[
1 = A(x + 5) + B(x + 1)
\]

\[
A = \frac{1}{4}, \quad B = -\frac{1}{4}
\]

Plugging this into our integrals gives us

\[
\int \frac{dx}{x^2 + 6x + 5} = \frac{1}{4} \ln(x + 1) - \frac{1}{4} \ln(x + 5)
\]

\[
= \frac{1}{4} \ln \left| \frac{x + 1}{x + 5} \right| + C
\]

Finally, taking the limits in the two definite integrals yields

\[
\int_{-2}^{4} \frac{dx}{x^2 + 6x + 5} = \lim_{N \to -1} \int_{-2}^{N} \frac{dx}{x^2 + 6x + 5} + \lim_{M \to -1} \int_{M}^{4} \frac{dx}{x^2 + 6x + 5}
\]

\[
= \lim_{N \to -1} \frac{1}{4} \ln \left| \frac{x + 1}{x + 5} \right|_{-2}^{N} + \lim_{M \to -1} \frac{1}{4} \ln \left| \frac{x + 1}{x + 5} \right|_{M}^{4}
\]

\[
= \infty - \infty
\]

so the improper integral is undefined. Note that we can’t use L’Hôpital’s rule and some algebra to simplify \( \infty - \infty \) in this case since we have two different limits, rather than two different terms in a single limit.

2. Suppose \( a_{n+2} = \frac{3}{2} a_{n+1} - \frac{1}{2} a_n \), with \( a_0 = 1 \) and \( a_1 = 1/2 \). Find \( \lim_{n \to \infty} a_n \). For partial credit you can just compute the first several terms and guess the limit. For full credit compute the first several terms, guess the formula for \( a_n \), show the formula satisfies the recurrence relation, and evaluate the limit using the formula.
Well, both approaches start with computing the first several terms:

\[
\begin{align*}
a_2 &= \frac{3}{2}a_1 - \frac{1}{2}a_0 = \frac{3}{4} - \frac{1}{2} = \frac{1}{4} \\
a_3 &= \frac{3}{2}a_2 - \frac{1}{2}a_1 = \frac{3}{8} - \frac{1}{4} = \frac{1}{8} \\
a_4 &= \frac{3}{2}a_3 - \frac{1}{2}a_1 = \frac{3}{16} - \frac{1}{8} = \frac{1}{16}
\end{align*}
\]

... 

Well it sure looks like the pattern is \( a_n = \frac{1}{2^n} \). Now we test this formula by checking that it satisfies the initial values and also the recurrence relation. \( a_0 = \frac{1}{2^0} = 1 \) and \( a_1 = \frac{1}{2^1} = 1/2 \) so the formula does have the right initial values. And

\[
\frac{3}{2} \left( \frac{1}{2^{n+1}} \right) - \frac{1}{2} \left( \frac{1}{2^n} \right) = \frac{3}{2^{n+2}} - \frac{1}{2^{n+1}} = \frac{3-2}{2^{n+2}} = \frac{1}{2^{n+2}}
\]

so the formula also satisfies the recurrence relation, so we have found the correct formula for the sequence. Then \( \lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{1}{2^n} = 0 \).

3. Evaluate \( \sum_{n=0}^{\infty} \frac{2}{n^2 + 6n + 8} \).

Since this isn’t a geometric series and the only other tool we have (so far) for evaluating series is to rewrite them in telescoping form, we use partial fractions to try to simplify the problem.

\[
\frac{2}{n^2 + 6n + 8} = \frac{A}{n+2} + \frac{B}{n+4}
\]

\[
2 = A(n+4) + B(n+2)
\]

\[
A = 1, \quad B = -1.
\]

Now we can rewrite our sum and check that it telescopes after the first two terms.

\[
\sum_{n=0}^{\infty} \frac{2}{n^2 + 6n + 8} = \sum_{n=0}^{\infty} \frac{1}{n+2} - \frac{1}{n+4}
\]

\[
= \left( \frac{1}{2} - \frac{1}{4} \right) + \left( \frac{1}{3} - \frac{1}{5} \right) + \left( \frac{1}{4} - \frac{1}{6} \right) + \left( \frac{1}{5} - \frac{1}{7} \right) + \ldots
\]

\[
= \frac{1}{2} + \frac{1}{3} + \left( -\frac{1}{4} + \frac{1}{4} \right) + \left( -\frac{1}{5} + \frac{1}{5} \right) + \ldots
\]

\[
= \frac{5}{6}.
\]
4. Evaluate \( \frac{2}{3} + \frac{1}{6} + \frac{1}{24} + \frac{1}{96} + \ldots \).

First we look at the pattern of the individual terms. We observe that each term is \( \frac{1}{4} \) the previous term (e.g. \( \frac{1}{6} = \frac{1}{4} \times \frac{1}{2} \), \( \frac{1}{24} = \frac{1}{4} \times \frac{1}{6} \), etc.). So we can recognize this as a geometric series with initial term=2/3, so the sum is \( \sum_{n=0}^{\infty} \frac{2}{3} \left( \frac{1}{4} \right)^n \). Then using the formula for sum of a geometric series we get the sum is \( \frac{2/3}{1-1/4} = \frac{2/3}{3/4} = \frac{8}{9} \).

5. Does \( \sum_{n=0}^{\infty} \frac{2n+3}{n^2 + 4n + 1} \) converge conditionally, converge absolutely, or diverge?

To determine convergence we have to worry about how things behave for large \( n \). For large \( n \), the rational function is dominated by the leading terms, so \( \frac{2n+3}{n^2 + 4n + 1} \approx \frac{2n}{n^2} = \frac{2}{n} \). This is nice because it is easy to see that \( \sum_{n} \frac{2}{n} \) diverges (it is just 2 times the harmonic series). So we can apply limit comparison between

\[
\sum_{n=0}^{\infty} \frac{2n+3}{n^2 + 4n + 1} \quad \text{and} \quad \sum_{n} \frac{2}{n}.
\]

We check that \( \lim_{n \to \infty} \frac{\frac{2n+3}{n^2 + 4n + 1}}{\frac{2}{n}} = \lim_{n \to \infty} \frac{2n^2 + 3n}{2n^2 + 8n + 2} = 1 \), which is positive and finite, and then since \( \sum_{n} \frac{2}{n} \) diverges, we can conclude by the limit comparison test that \( \sum_{n=0}^{\infty} \frac{2n+3}{n^2 + 4n + 1} \) diverges.

6. What is the 2nd degree Taylor polynomial approximation for \( x \sin(x) \) about \( x = \pi \)?

This looks like a problem we can do quickly using the known Maclaurin series for \( \sin(x) \) (recall the Maclaurin series is just the Taylor series about \( x_0 = 0 \)). Unfortunately, we are asked for the approximation about \( x = \pi \), and since that is a different center point, we will have to work this out from scratch.

\[
f(x) = x \sin(x), \quad f(\pi) = \pi \sin(\pi) = 0
\]

\[
f'(x) = \sin(x) + x \cos(x), \quad f'(\pi) = \sin(\pi) + \pi \cos(\pi) = 0 - \pi = -\pi
\]

\[
f''(x) = 2 \cos(x) - x \sin(x), \quad f''(\pi) = 2 \cos(\pi) - \pi \sin(\pi) = -2 - 0 = -2
\]

So the 2nd degree Taylor polynomial approximation is

\[
-\pi(x-\pi) + \frac{-2}{2!}(x-\pi)^2 = -\pi(x-\pi) - (x-\pi)^2.
\]
7. What is the radius of convergence of \( \sum_{n=0}^{\infty} \frac{n(x-1)^n}{3^n} \)?

The radius of convergence is usually easiest to determine using the ratio test. In this series, we have the terms \( a_n = \frac{n(x-1)^n}{3^n} \), so we form the ratio

\[
\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{(n+1)(x-1)^{n+1}}{n(x-1)^n 3^{n+1}} \right| = \frac{(n+1)(x-1)^{n+1}3^n}{n(x-1)^n 3^{n+1}} = \frac{(n+1)|x-1|}{3n}.
\]

Then we take the limit and find

\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \frac{(n+1)|x-1|}{3n} = \frac{|x-1|}{3}.
\]

Now this limit is < 1, hence the series converges, when \(|x-1| < 3\), and the limit is > 1, hence the series diverges, when \(|x-1| > 3\). So the radius of convergence is 3. (Note: the series diverges at both endpoints by the \( n^{th} \)-term test, but that isn’t asked for in the problem).

8. What is an upper bound for the error in the approximation \( \cos(2x) \approx 1 - 2x^2 + \frac{2x^4}{3} \) at the point \( x = 2 \)? You must justify the upper bound from the error formula, not just evaluate the difference in the two values by trusting your calculator.

The approximation given is the 4\(^{th}\) degree Taylor polynomial approximation about \( x_0 = 0 \). So we find the error using the remainder formula for Taylor polynomials,

\[
|R_4(2)| = \left| \frac{-32\sin(2z)}{5!} (2-0)^5 \right| = \frac{1024}{120} |\sin(2z)|
\]

where \( z \) lies between 0 and 2. Here we have used the fact that the 5\(^{th}\) derivative of \( \cos(2x) \) is \(-32\sin(2x)\). Now the largest \( |\sin(2z)| \) can be is 1, so the largest the error could be is \( \frac{1024}{120} = \frac{128}{15} \approx 8.53 \).

9. A child drops a super ball on a hard floor from a distance of 1 meter. Each time the ball bounces, it comes back up to 80% of its previous height. The first time the ball fails to bounce 10cm high, the child grabs the ball. Assuming the ball’s motion was straight up and down, how far did the ball travel?

First the ball falls 1 meter Distance traveled: 1 m
Then the ball bounces up and down .8 m Distance traveled: 1.6 m
Then the ball bounces up and down .64m Distance traveled: 1.28 m

... Then the ball bounces up and down .8\(^n\) m Distance traveled: 2*0.8\(^n\) m

... Then the ball bounces up .8\(^{m}\) m and is caught since .8\(^{m}\) < .1 Distance traveled: .8\(^{m}\) m
So the formula for the typical distance traveled in a bounce is total distance traveled is \(2 \cdot 0.8^n\) m and the total distance traveled is then \(2 \sum_{n=0}^{m} 0.8^n - 1 - 0.8^m\) meters, where we subtract off the two terms where the ball only travels up or down and not both ways. Using the formula for a finite geometric series, the total distance is then

\[
2 \sum_{n=0}^{m} 0.8^n - 1 - 0.8^m = 2 \left( \frac{1 - 0.8^{m+1}}{1 - 0.8} \right) - 1 - 0.8^m \\
= 2 \left( \frac{1 - 0.8^{m+1}}{1 - 0.8} \right) - 1 - 0.8^m \\
= 10 - 10(0.8^{m+1}) - 1 - 0.8^m \\
= 10 - 8(0.8^m) - 1 - 0.8^m \\
= 9 - 9(0.8^m).
\]

From the definition of \(m\) as the first time \(0.8^m < 0.1\), we can see our answer will be just over 8.1 meters. To find the exact value we need to find the exact value of \(m\). We can do this fairly easily just by guess and check with a calculator, but we can also do it analytically using logarithms. If \(0.8^x = 0.1\), then \(x \log(0.8) = \log(0.1)\), so \(x \approx 10.32\) and hence the first integer \(m\) with \(0.8^m < 0.1\) is \(m = 11\). So the final answer is \(9 - 9(0.8^{11}) \approx 8.2\) m.
10. Match the following sequences with their graphs.

a. \( a_n = \frac{5^n}{n!} \)

b. \( b_n = \frac{n^2 + 1}{n^3 + 1} \)

c. \( c_n = \sum_{k=0}^{n} \frac{2}{3^k} \)

d. \( d_n = \sum_{k=0}^{n} \frac{1}{k^2 + 3k + 2} \)