Math 221 – Fall 2004
Practice Exam 3

75 minutes. Closed book. You are allowed a calculator and one 8.5” x 11” sheet of handwritten notes. Not all problems on Exam 3 will have a matching problem on this practice test, but this practice test should give you a sense of what the exam will be like as you study. You will be provided with a table of integrals for the exam. You will need to show all your work to receive full credit.

1. Evaluate \( \lim_{x \to 0} \frac{2^{(2^x)} - 4}{x} \).

Substituting \( x = 0 \) into the equation, we have a limit of the form \(-2/0\), which is undefined (it doesn’t tend to \( \pm \infty \) since the values can be either positive or negative depending on whether \( x < 0 \) or \( x > 0 \)). **Note:** You didn’t need to differentiate \( 2^{(2^x)} \), but you could do this if you needed to by remembering that the derivative of \( a^x \) is \( \ln(a)a^x \) and using the chain rule to get \( \ln(2)2^{(2^x)}\ln(2)2^x = 2^{(2^x)}2^x \ln(2)^2 \).

2. Evaluate \( \lim_{x \to 0} \frac{\cos(x) + \sin(x) - 2}{x} \).

Substituting \( x = 0 \) into the equation, we have a limit of the form \(-1/0\), which is undefined (it doesn’t tend to \( \pm \infty \) since the values can be either positive or negative depending on whether \( x < 0 \) or \( x > 0 \)). **Note:** I didn’t intend both of these to be problems where L’Hôpital’s rule didn’t apply, but I made a typo in one problem. Of course, there might also be a typo on the test. Always check the L’Hôpital’s rule applies before using it.

3. Evaluate \( \int_{2}^{4} \frac{dx}{x^2\sqrt{4x^2 - 9}} \). You need to give the exact answer, not a decimal approximation.

**This problem also had a typo when first posted and has been corrected (the lower limit needed to be 2 instead of 1 or the integrand was undefined for part of the range of integration).** Looking at the table of integrals, this looks similar to #35. We make the substitution \( u = 2x \) to get the radical in the right form. Then \( du = 2dx \) and so we multiply and divide by 2 to the integral in the appropriate form, then read the answer from the table (remembering to change the limits as well as the integrand when we make our substitution).
\[
\int_{\frac{4}{2}} dx = 2 \int_{4}^{2} dx = 2 \int_{0}^{1} \frac{2}{(2x)^2} \sqrt{(2x)^2 - 9} \quad du = 2 \int_{0}^{4} u^2 \sqrt{u^2 - 3^2} \quad \Rightarrow \\
\frac{\sqrt{u^2 - 3^2}}{3^2u} \bigg|_{0}^{4} = \frac{\sqrt{55} - \sqrt{7}}{18} = \frac{\sqrt{55} - 2\sqrt{7}}{36} = \frac{\sqrt{55} - \sqrt{28}}{36}.
\]

Any of the answers on the last line are acceptable, but remember that \(\sqrt{55} - \sqrt{28} \neq \sqrt{27}\).

4. Evaluate \(\int_{\frac{4}{2}}^{\frac{1}{2}} \frac{dx}{x^2 + 6x + 5}\). You need to give the exact answer, not a decimal approximation.

This is an improper integral, since it has a singularity at \(x = -1\). So we rewrite it as a pair of limits.

\[
\int_{\frac{4}{2}}^{\frac{1}{2}} \frac{dx}{x^2 + 6x + 5} = \lim_{N \to -1} \int_{\frac{4}{2}}^{N} \frac{dx}{x^2 + 6x + 5} + \lim_{M \to -1} \int_{M}^{\frac{1}{2}} \frac{dx}{x^2 + 6x + 5}
\]

Now we evaluate the integrals using partial fractions.

\[
\frac{1}{x^2 + 6x + 5} = \frac{A}{(x + 1)} + \frac{B}{(x + 5)}
\]

\(1 = A(x + 5) + B(x + 1)\)

\(A = 1/4, \quad B = -1/4\)

Plugging this into our integrals gives us

\[
\int \frac{dx}{x^2 + 6x + 5} = \frac{1}{4} \ln(x + 1) - \frac{1}{4} \ln(x + 5)
\]

\(= \frac{1}{4} \ln \left| \frac{x + 1}{x + 5} \right| + C\)

Finally, taking the limits in the two definite integrals yields

\[
\int_{\frac{4}{2}}^{\frac{1}{2}} \frac{dx}{x^2 + 6x + 5} = \lim_{N \to -1} \int_{\frac{4}{2}}^{N} \frac{dx}{x^2 + 6x + 5} + \lim_{M \to -1} \int_{M}^{\frac{1}{2}} \frac{dx}{x^2 + 6x + 5}
\]

\[= \lim_{N \to -1} \frac{1}{4} \ln \left| \frac{x + 1}{x + 5} \right|_{0}^{N} + \lim_{M \to -1} \frac{1}{4} \ln \left| \frac{x + 1}{x + 5} \right|_{0}^{M}
\]

\[= \infty - \infty\]
so the improper integral is **undefined**. Note that we can’t use L’Hôpital’s rule and some algebra to simplify \( \infty - \infty \) in this case since we have two different limits, rather than two different terms in a single limit.

5. Evaluate \( \int_0^\infty x^2 e^{-2x} \, dx \). You need to give the exact answer, not a decimal approximation.

This is an improper integral since one of the limits is \( \infty \). We rewrite the integral as a limit and evaluate using integration by parts.

\[
\int_0^\infty x^2 e^{-2x} \, dx = \lim_{N \to \infty} \int_0^N x^2 e^{-2x} \, dx \\
= \lim_{N \to \infty} \left[ -\frac{1}{2} x^2 e^{-2x} - \frac{1}{2} xe^{-2x} - \frac{1}{4} e^{-2x} \right]_0^N \\
= \frac{1}{4}.
\]

6. Evaluate \( \int_0^{\ln 2} \sqrt{4 - e^{2x}} \, e^x \, dx \). You need to give the exact answer, not a decimal approximation.

This problem doesn’t immediately look like anything we have done or in the tables. Since the sticking point is all the exponentials, we try making the substitution \( u = e^x \), so \( du = e^x \, dx \) and we get

\[
\int_{x=0}^{\ln 2} \sqrt{4 - e^{2x}} \, e^x \, dx = \int_{u=2}^{u=4} \sqrt{4 - u^2} \, du \\
= \frac{1}{2} \left( u \sqrt{4 - u^2} + 4 \arcsin \left( \frac{u}{2} \right) \right) \bigg|_2^4 \quad \text{(using \#37 from the table of integrals)} \\
= 2 \arcsin(1) - \left( \frac{\sqrt{3}}{2} + 2 \arcsin \left( \frac{1}{2} \right) \right).
\]

7. Suppose \( a_{n+2} = \frac{3}{2} a_{n+1} - \frac{1}{2} a_n \), with \( a_0 = 1 \) and \( a_1 = 1/2 \). Find \( \lim_{n \to \infty} a_n \). For partial credit you can just compute the first several terms and guess the limit. For full credit compute the first several terms, guess the formula for \( a_n \), show the formula satisfies the recurrence relation, and evaluate the limit using the formula.

Well, both approaches start with computing the first several terms:
\[ a_2 = \frac{3}{2}a_1 - \frac{1}{2}a_0 = \frac{3}{4} - \frac{1}{4} = \frac{1}{4} \]
\[ a_3 = \frac{3}{2}a_2 - \frac{1}{2}a_1 = \frac{3}{8} - \frac{1}{4} = \frac{1}{8} \]
\[ a_4 = \frac{3}{2}a_3 - \frac{1}{2}a_2 = \frac{3}{16} - \frac{1}{8} = \frac{1}{16} \]

Well it sure looks like the pattern is \( a_n = \frac{1}{2^n} \). Now we test this formula by checking that it satisfies the initial values and also the recurrence relation. \( a_0 = \frac{1}{2^0} = 1 \) and \( a_1 = \frac{1}{2^1} = \frac{1}{2} \) so the formula does have the right initial values. And

\[
\frac{3}{2} \left( \frac{1}{2^{n+1}} \right) - \frac{1}{2} \left( \frac{1}{2^n} \right) = \frac{3}{2^{n+1}} - \frac{1}{2^{n+1}} = \frac{3 - 2}{2^{n+1}} = \frac{1}{2^{n+1}}
\]

so the formula also satisfies the recurrence relation, so we have found the correct formula for the sequence. Then \( \lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{1}{2^n} = 0 \).

8. Evaluate \( \sum_{n=0}^\infty \frac{2}{n^2 + 6n + 8} \).

Since this isn’t a geometric series and the only other tool we have (so far) for evaluating series is to rewrite them in telescoping form, we use partial fractions to try to simplify the problem.

\[
\frac{2}{n^2 + 6n + 8} = \frac{A}{n+2} + \frac{B}{n+4}
\]

\[
2 = A(n+4) + B(n+2)
\]

\( A = 1, \quad B = -1. \)

Now we can rewrite our sum and check that it telescopes after the first two terms.

\[
\sum_{n=0}^\infty \frac{2}{n^2 + 6n + 8} = \sum_{n=0}^\infty \frac{1}{n+2} - \frac{1}{n+4}
\]

\[
= \left( \frac{1}{2} - \frac{1}{4} \right) + \left( \frac{1}{3} - \frac{1}{5} \right) + \left( \frac{1}{4} - \frac{1}{6} \right) + \left( \frac{1}{5} - \frac{1}{7} \right) + \ldots
\]

\[
= \frac{1}{2} + \frac{1}{3} + \left( -\frac{1}{4} + \frac{1}{4} \right) + \left( -\frac{1}{5} + \frac{1}{5} \right) + \ldots
\]

\[
= \frac{5}{6}.
\]

9. A child drops a super ball on a hard floor from a distance of 1 meter. Each time the ball bounces, it comes back up to 80% of its previous height. The first time the ball
fails to bounce 10cm high, the child grabs the ball. Assuming the ball’s motion was straight up and down, how far did the ball travel?

First the ball falls 1 meter
Then the ball bounces up and down .8 m
Then the ball bounces up and down .64 m

\[ \text{Distance traveled: } 1 \text{ m} \]
\[ \text{Distance traveled: } 1.6 \text{ m} \]
\[ \text{Distance traveled: } 1.28 \text{ m} \]

\[ \ldots \]
Then the ball bounces up and down \(.8^n\) m
\[ \text{Distance traveled: } 2 \cdot .8^n \text{ m} \]
\[ \ldots \]
Then the ball bounces up \(.8^m\) m and is caught since \(.8^m < .1\)

\[ \text{Distance traveled: } .8^m \text{ m} \]

So the formula for the typical distance traveled in a bounce is total distance traveled is \(2 \cdot .8^n\) m and the total distance traveled is then \(2 \sum_{n=0}^{m} .8^n - 1 - .8^m\) meters, where we subtract off the two terms where the ball only travels up or down and not both ways. Using the formula for a finite geometric series, the total distance is then

\[
2 \sum_{n=0}^{m} .8^n - 1 - .8^m = 2 \frac{1 - .8^{m+1}}{1 - .8} - 1 - .8^m
\]

\[
= 10 - 10(.8^{m+1}) - 1 - .8^m
\]

\[
= 10 - 8(.8^m) - 1 - .8^m
\]

\[
= 9 - 9(.8^m).
\]

From the definition of \(m\) as the first time \(.8^m < 0.1\), we can see our answer will be just over 8.3 meters. To find the exact value we need to find the exact value of \(m\). We can do this fairly easily just by guess and check with a calculator, but we can also do it analytically using logarithms. If \(0.8^x = 0.1\), then \(x \log(0.8) = \log(0.1)\), so \(x \approx 10.32\) and hence the first integer \(m\) with \(.8^m < 0.1\) is \(m = 11\). So the final answer is \(9 - 9(.8^{11}) \approx 8.2\) m.
10. Match the following sequences with their graphs.

a. \( a_n = \frac{5^n}{n!} \)

b. \( b_n = \frac{n^2 + 1}{n^3 + 1} \)

c. \( c_n = \sum_{k=0}^{n} \frac{2}{3^k} \)

d. \( d_n = \sum_{k=0}^{n} \frac{1}{k^2 + 3k + 2} \)