Show all work for full credit. You may not use notes or books. You may use calculators (such as the TI-83, TI-84, TI-85 or TI-86) to check your work and help with arithmetic. You may not use the TI-89 or higher.

(10 pts) 1. Let \( f''(x) = e^x + 2x \), \( f(0) = 0 \) and \( f(1) = e + 1 \). Find \( f(x) \).

\[
\begin{align*}
    f'(x) &= e^x + x^2 + c \\
    f(x) &= e^x + \frac{x^3}{3} + cx + d \\
    f(0) &= e^0 + 0 + 0 + d = 1 + d = 0 \quad \therefore d = -1 \\
    f(1) &= e^1 + \frac{1}{3} + c + d = e + \frac{1}{3} + c - 1 = e + 1 \\
    f(x) &= e^x + \frac{x^3}{3} + \frac{5}{3}x - 1
\end{align*}
\]

(10 pts) 2. Find the absolute maximum and absolute minimum values of \( f(x) = x^3 - 3x + 1 \) on \([0,2]\).

\[
\begin{align*}
    f''(x) &= 3x^2 - 3 = 0 \\
    3(x^2 - 1) &= 3(x+1)(x-1) = 0 \\
    \text{critical numbers:} \quad x = -1, \quad x = 1 \\
    f(0) &= 0^3 - 3\cdot0 + 1 = 1 \\
    f(1) &= 1^3 - 3\cdot1 + 1 = -1 \quad \text{abs min} \\
    f(2) &= 2^3 - 3\cdot2 + 1 = 3 \quad \text{abs max}
\end{align*}
\]

\( f \) has an abs max of 3 at \( x = 2 \).

\( f \) has an abs min of -1 at \( x = 1 \).
(30 pts) 3. Evaluate the integrals.

(a) \[ \int_1^9 \frac{1}{x} \, dx = \left| \ln x \right| \bigg|_1^9 = \left| \ln 9 \right| - \left| \ln 1 \right| = \ln 9 - \ln 1 \]

This is acceptable.

(b) \[ \int_0^{\pi/2} \cos \theta (\sin^2 \theta + 2) \, d\theta = \int_0^{\pi/2} \sin^2 \theta \cos \theta \, d\theta + \int_0^{\pi/2} 2 \cos \theta \, d\theta \]

For the first integral: \( u = \sin \theta \quad du = \cos \theta \, d\theta \)

\[ u(0) = 0 \quad u(\pi/2) = 1 \]

\[ = \int_0^{\pi/2} u^2 \, du + 2 \int_0^{\pi/2} \cos \theta \, d\theta = \frac{u^3}{3} \bigg|_0^{\pi/2} + 2 \sin \theta \bigg|_0^{\pi/2} \]

\[ = \left( \frac{1}{3} - \frac{1}{3} \right) + 2 \left( \sin \frac{\pi}{2} - \sin 0 \right) = \frac{1}{3} + 2 = \frac{7}{3} \]

(c) \[ \int_1^3 \frac{2 + \sqrt{x}}{\sqrt{x}} \, dx \]

\[ u = 2 + \sqrt{x} \quad du = \frac{1}{2} \sqrt{x} \, dx \]

\[ u(1) = 2 + \sqrt{1} = 3 \quad u(3) = 2 + \sqrt{3} \]

\[ 2 \, du = \frac{1}{\sqrt{x}} \, dx \]

\[ \int_1^3 \frac{2 + \sqrt{x}}{\sqrt{x}} \, dx \]

\[ = \int_3^{2+\sqrt{3}} u \cdot 2 \, du \]

\[ = \left. \frac{u^2}{2} \right|_3^{2+\sqrt{3}} = \left( 2 + \sqrt{3} \right)^2 - 3^2 = 4 + 4\sqrt{3} + 3 - 9 \]

This is acceptable.

\[ = 4\sqrt{3} - 2 \]
(d) \[ \int \sqrt{2-t} \, dt \]
\[ u = 2-t \]
\[ \frac{du}{dt} = -1 \]
\[ -du = dt \]
\[ \int \sqrt{u} \, (-du) = \int u^{\frac{1}{2}} \, du \]
\[ = -\frac{u^{\frac{3}{2}}}{\frac{3}{2}} + C \]
\[ = -\frac{2}{3}(2-t)^{\frac{3}{2}} + C \]

(e) \[ \int \frac{x^2 + \frac{1}{3}}{x^3 + 3x} \, dx \]
\[ u = x^3 + 3x \]
\[ \frac{du}{dx} = 3x^2 + 3 = 3(x^2 + 1) \]
\[ \frac{1}{3} \, du = (x^2 + 1) \, dx \]
\[ \int \frac{x^2 + \frac{1}{3}}{x^3 + 3x} \, dx = \frac{1}{3} \int \frac{1}{u} \, du \]
\[ = \frac{1}{3} \ln |u| + C \]
\[ = \frac{1}{3} \ln |x^3 + 3x| + C \]

(f) \[ \int xe^{2x^2} \, dx \]
\[ u = 2x^2 \]
\[ \frac{du}{dx} = 4x \]
\[ \frac{1}{4} \, du = x \, dx \]
\[ \int xe^{2x^2} \, dx = \frac{1}{4} \int e^u \, du = \frac{1}{4} e^u + C \]
\[ = \frac{1}{4} e^{2x^2} + C \]
(10 pts) 4. The graph of $y = f(x)$ is shown in the picture.

(a) Use three rectangles to find estimates of each type for the area under the given graph of $f$ from $x = 0$ to $x = 6$.

(i) $L_3$ (use left endpoints).

\[ L_3 = 3 \cdot 2 + 3 \cdot 2 + 7 \cdot 2 = 6 + 6 + 14 = \boxed{26} \]

(ii) $R_3$ (use right endpoints).

\[ R_3 = 3 \cdot 2 + 7 \cdot 2 + 5 \cdot 2 = 6 + 14 + 10 = \boxed{30} \]

(iii) $M_3$ (use midpoints).

\[ M_3 = 2 \cdot 3 + 2.5 + 2.6 = 6 + 10 + 12 = \boxed{28} \]

(b) Explain which of the numbers $L_3, R_3, M_3$ gives the best estimate of the area under $y = f(x)$.

$M_3$ is the area under $y = f(x)$. Compare the areas of the shaded triangles above.
(10 pts) 5. Sketch the region enclosed by the curves \( y = x^2 - 3 \) and \( y = -2x^2 \) and find the area of the region. Sketch an approximating rectangle and label its height and width. Please label the graph clearly.

First, find where the curves intersect.

\[
\begin{align*}
X^2 - 3 &= -2X^2 \\
3X^2 - 3 &= 0 \\
3(X^2 - 1) &= 0 \\
3(X+1)(X-1) &= 0 \\
X &= -1, X = 1
\end{align*}
\]

Next, sketch the graph.

\[
\begin{array}{c|ccc}
X & X^2 - 3 & -2X^2 \\
-1 & -2 & -2 \\
0 & -3 & 0 \\
1 & -2 & -2 \\
\end{array}
\]

\[
A = \int_{-1}^{1} (-2x - (x^2 - 3)) \, dx = \int_{-1}^{1} (-3x^2 + 3) \, dx = \left[ -\frac{3x^3}{3} + 3x \right]_{-1}^{1} \\
\text{(Notice } A = 2 \int_{0}^{1} (-5x^2 - 3) \, dx \text{ would be easier.)} \\
\text{Doing that is okay since the region is symmetrical about the y-axis.}
\]

\[
\begin{align*}
= & \left[ -x^3 + 3x \right]_{-1}^{1} \\
= & (-1 + 3) - (-(-1) - 3) \\
= & 2 - (-2) = 4
\end{align*}
\]
(10 pts) 6. Find the volume of the solid obtained by rotating the region bounded by the curves \( y = e^x, \ y = 0, \ x = 0, \) and \( x = 2 \) about the \( x \)-axis. Sketch the region and the solid. Please label the graph clearly. Please give the exact answer for volume.

First, graph the region.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( e^x )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>( e )</td>
</tr>
<tr>
<td>2</td>
<td>( e^2 )</td>
</tr>
</tbody>
</table>

Then rotate the region about the \( x \)-axis.

Area of disk \( A(x) = \pi r^2 = \pi (e^x)^2 = \pi e^{2x} \)

\[
V = \int_0^2 \pi e^{2x} \, dx = \pi \left[ \frac{1}{2} e^{2x} \right]_0^2 = \pi \left( \frac{1}{2} e^4 - \frac{1}{2} e^0 \right) = \frac{\pi}{2} (e^4 - 1)
\]
(10 pts) 7(a) Let \( f(x) = 3\sqrt{x} \). Find the average value of the function on \([0, 4]\).

\[
\bar{f} = \frac{1}{b-a} \int_a^b f(x) \, dx = \frac{1}{4} \int_0^4 3\sqrt{x} \, dx
\]

\[
= \frac{3}{4} \left[ \frac{2}{3} x^{3/2} \right]_0^4
\]

\[
= \frac{1}{2} \cdot 4^{3/2} = \frac{1}{2} \cdot (4)^{3/2} = 4
\]

(b) Find \( c \) such that \( \bar{f} = f(c) \), (\( \bar{f} \) is the number you found in part (a)).

\[
4 = f(c)
\]

\[
4 = 3\sqrt{c}
\]

\[
\frac{4}{3} = \sqrt{c}
\]

\[
\left(\frac{4}{3}\right)^2 = c
\]

\[
c = \frac{16}{9}
\]

(c) Sketch the graph of \( f \) and a rectangle whose area is the same as the area under the graph of \( f \). Please label the graph clearly.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( 3\sqrt{x} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>3\sqrt{2}</td>
</tr>
<tr>
<td>3</td>
<td>3\sqrt{3}</td>
</tr>
<tr>
<td>4</td>
<td>6</td>
</tr>
</tbody>
</table>

Area under \( y = f(x) \) is \( \int_0^4 f(x) \, dx = 4 \cdot 4 = 16 \)
(21 pts) 8. Evaluate the following limits. If the limit does not exist, explain why.

(a) \( \lim_{x \to 4} 3 + \frac{2x}{9} = 3 + \frac{2(4)}{9} = \frac{38}{9} \)

(b) \( \lim_{x \to 4} \frac{x^2 + x - 12}{x + 4} = \lim_{x \to -4} \frac{(x+4)(x-3)}{x+4} = \lim_{x \to 4} (x-3) = -7 \)

So \( \lim_{x \to 4} \frac{x^2 + x - 12}{x + 4} = \lim_{x \to 4} \frac{2x+1}{1} = -7 \)

(c) \( \lim_{x \to 7} \frac{x^2 - 49}{\sqrt{x} - \sqrt{7}} = \lim_{x \to 7} \frac{(x+7)(x-7)}{\sqrt{x} - \sqrt{7}} = \lim_{x \to 7} \frac{(x+7)(\sqrt{x} + \sqrt{7})(\sqrt{x} - \sqrt{7})}{\sqrt{x} - \sqrt{7}} = \lim_{x \to 7} (x + 7)(\sqrt{x} + \sqrt{7}) = 14(2\sqrt{7}) = 28\sqrt{7} \)

(d) \( \lim_{x \to 0} \frac{|x|}{x} \)

\( \lim_{x \to 0^+} \frac{|x|}{x} = \lim_{x \to 0^+} \frac{x}{x} = \lim_{x \to 0^+} 1 = 1 \)

\( \lim_{x \to 0^-} \frac{|x|}{x} = \lim_{x \to 0^-} \frac{-x}{x} = \lim_{x \to 0^-} -1 = -1 \)

The left and right limits aren't equal, so \( \lim_{x \to 0} \frac{|x|}{x} \) doesn't exist.
(e) \( \lim_{x \to 0} \frac{3x}{\sin \pi x} \)  
Form: \( \frac{0}{0} \)

\[
\frac{d}{dx} \left( \frac{3}{\pi \cos \pi x} \right) = \frac{3}{\pi} \cdot \frac{\pi \cdot 1}{\cos \pi x} = \frac{3}{\pi}
\]

(f) \( \lim_{x \to 1} \frac{x^{14} - 1}{x^3 - 1} \)  
Form: \( \frac{0}{0} \)

\[
\frac{d}{dx} \left( \frac{14x^{13}}{3x^2} \right) = \frac{14x^{13} \cdot 0}{3(1)} = \frac{14}{3}
\]

(g) \( \lim_{x \to 5^+} \frac{2}{x - 5} \)  
Form: \( \frac{\text{constant}^+}{0} \). Not a form that L'Hopital's rule applies. It's an infinite limit.

\[
\lim_{x \to 5^+} \frac{2}{x - 5} = +\infty
\]
(10 pts) 9. Evaluate \( \int_0^6 f(x) \, dx \) by interpreting it in terms of areas, where

\[
f(x) = \begin{cases} 
|x - 2|, & \text{if } 0 \leq x \leq 4 \\ 
2, & \text{if } 4 < x \leq 6
\end{cases}
\]

Hint: Sketch the graph.

\[
\int_0^6 f(x) \, dx = A(x) = \frac{1}{2} \cdot 2 + \frac{1}{2} \cdot 2 + 2 \cdot 2
\]

\[= \frac{1}{2} (2) + \frac{1}{2} (2) + 2 \cdot 2 \]

\[= 2 + 2 + 4 = 8\]
(9 pts) 10. The graph of $y = f(x)$ is shown in the picture.

(a) State the intervals on which $f$ is continuous.
\[ (-\infty, 3) \cup (3, 5) \cup (5, \infty) \]

(b) Find, or state that it does not exist.
\[
\lim_{x \to 5^{-}} f(x) = \text{?} \\
\lim_{x \to 5} f(x) = \text{DNF}
\]

(c) Is $f$ differentiable at $x = 1$? Yes \quad x = 4? \quad No \quad x = 5? \quad No

(d) Find:
\[ f'(3.5) = 2 \quad f''(3.5) = 0 \]

(e) Find the limit \[ \lim_{h \to 0} \frac{f(1 + h) - f(1)}{h} = 0 \]

(f) Does \[ \lim_{h \to 0} \frac{f(4 + h) - f(4)}{h} \] exist? Explain. No.

To the left of $x = 4$, the slope of $f(x)$ is 2.

To the right of $x = 4$, the slope of $f(x)$ is -1.

These must be equal for $f'(4)$ to exist.
(10 pts) 11. Use logarithmic differentiation to find the derivative of the function 
\( y = x^{2x} \).

\[
\begin{align*}
\ln y & = \ln x \cdot x^2 = 2x \ln x \\
\frac{1}{y} \frac{dy}{dx} & = 2x \cdot \frac{1}{x} + 2 \ln x \\
\frac{dy}{dx} & = y \left( 2 + 2 \ln x \right) \\
\frac{dy}{dx} & = x^{2x} \left( 2 + 2 \ln x \right)
\end{align*}
\]

(10 pts) 12. Find \( \frac{dy}{dx} \) by implicit differentiation: \( x^3 - 2xy + y^2 = 5 \).

\[
\begin{align*}
3x^2 - 2x \frac{dy}{dx} - 2y + 2y \frac{dy}{dx} & = 0 \\
3x^2 - 2y & = 2x \frac{dy}{dx} - 2y \frac{dy}{dx} = (2x - 2y) \frac{dy}{dx}
\end{align*}
\]

\[
\frac{dy}{dx} = \frac{3x^2 - 2y}{2x - 2y}
\]

(10 pts) 13. Find the equation of the line tangent to the curve \( y = 2x^2 + \sin 2x + 3 \) at (0, 3).

\[
\begin{align*}
\gamma' & = 4x + 2 \cos 2x \\
m & = \gamma'(0) = 4(0) + 2 \cos (0) = 2
\end{align*}
\]

\[
\gamma = 2x + 3
\]
(10 pts) 14. John rented 100 apartments for $400 monthly rent. He noticed that for each $20 increase in rent, 10 fewer apartments would be rented. What should he charge for rent to maximize revenue? Hint: First, find the price function $p(x)$. You may assume it is linear.

$$\begin{align*}
\text{units} & \quad \text{rent} \\
X_1 & \quad p(x) \\
(100, 400) & \quad m = \frac{420 - 400}{90 - 100} = \frac{20}{-10} = -2 \\
(90, 420) & \\
\gamma - 400 = -2(x - 100) \\
\gamma & = -2x + 200 + 400 \\
\gamma & = -2x + 600 \\
p(x) & = -2x + 600
\end{align*}$$

Revenue: $R(x) = xp(x) = -2x^2 + 600x$

$$R'(x) = -4x + 600 = 0$$

$4x = 600$

$x = 150$ units

$p(150) = -2(150) + 600 = $300 monthly rent.

Does this really maximize $R(x)$?

$R''(x) = -4$, so $R(x)$ is always concave down.

By the 2nd Der. Test for Abs. Extreme Values, $R(150)$ is the abs. max revenue.
(20 pts) 15. Differentiate the function. No simplification is required.

(a) \( y = \ln(x^2 \cos x) \)

\[
\frac{dy}{dx} = \frac{1}{x^2 \cos x} \left( x^2 \cos x \right)' = \frac{1}{x^2 \cos x} \left( x^2 (-\sin x) + 2x \cos x \right)
\]

\[
\frac{dy}{dx} = -x^2 \sin x + 2x \cos x
\]

\[
\therefore y = 2 \ln x + \ln \cos x
\]

(b) \( y = \frac{x^2 + 2}{7x - 1} \)

\[
\frac{dy}{dx} = \frac{(7x-1)(2x)-(x^2+2)(7)}{(7x-1)^2}
\]

\[
\frac{dy}{dx} = \frac{(x^2+2)(7x-1)^{-1}}
\]

\[
\therefore y = \frac{x^2+2}{7} - \frac{2x}{7} - \frac{x^2+2}{7x-1}
\]

(c) \( y = e^{x^2+3} + e^{-5x} \)

\[
\frac{dy}{dx} = 2xe^{x^2+3} - 5e^{-5x}
\]

(d) \( y = (x^2 + 3)(x^{2/3} - 5) \)

\[
\frac{dy}{dx} = (x^2 + 3) \left( \frac{2}{3} x^{-1/3} \right) + 2x \left( x^{2/3} - 5 \right)
\]